Master 1 — Spring 2025
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TD12 : Harmonic Functions and Miscellaneous

Exercise 1 — *Area of planar Brownian motion.*
Let B be a planar Brownian motion. let $\mathcal{L}^2$ denote the Lebesgue measure on $\mathbb{R}^2$. We are going to show that almost surely $\{B_t, t \in [0,1]\}$ is $\mathcal{L}^2$-negligible.

(1) Let $A_1, A_2 \subset \mathbb{R}^2$ be measurable sets, show that if $\mathcal{L}^2(A_1), \mathcal{L}^2(A_2) > 0$ then

$$\mathcal{L}^2\left\{x \in \mathbb{R}^2 : \mathcal{L}^2(A_1 \cap (x + A_2)) > 0\right\} > 0.$$

(2) Let $X_I = \mathcal{L}^2\{B_t, t \in I\}$, by bounding $\mathbb{P}(X_I > a)$, show that for every bounded interval $I \subset \mathbb{R}_+$, the expectation of X_I is finite.

(3) Show that $\mathcal{L}^2\{B_t, t \in [0,1]\} \cap \mathcal{L}^2\{B_t, t \in [2,3]\} = 0$ almsot surely.

(4) Let $B'_t = B_{t+2} - B_2 + B_1$ and

$$R(x) = \mathcal{L}^2\{B_t : t \in [0,1]\} \cap \{x + B'_t : t \in [0,1]\}.$$

Show that for almost all $x \in \mathbb{R}^2$, we have $\mathbb{P}(R(x) = 0) = 1$.

(5) Deduce from the previous question that almost surely,

$$\mathcal{L}^2\{x \in \mathbb{R}^2 : R(x) > 0\} = 0.$$

(6) Deduce from the previous questions that almost surely $\{B_t : t \in [0,1]\}$ is $\mathcal{L}^2$-negligible.

Exercise 2 — *Point transience of Brownian motion.*
Let $d \geq 2$ and B a Brownian motion on $\mathbb{R}^d$. We wish to show that for every $y \in \mathbb{R}^d$, we have

$$\mathbb{P}(y \in \{B_t, t \in (0,1]\}) = 0.$$

(1) Show that the result for $d = 2$ implies the result for all $d \geq 2$. In the rest of the exercise we assume that $d = 2$.

(2) Show for every $x \in \mathbb{R}^d$ and almost all $y \in \mathbb{R}^d$, $\mathbb{P}_x(y \in \{B_t, t \in (0,1]\}) = 0$.

(3) Deduce that $y \in \mathbb{R}^2$ and almost all $x \in \mathbb{R}^2$, $\mathbb{P}_x(y \in \{B_t, t \in (0,1]\}) = 0$.

(4) Conclude. (Hint: you may consider $\mathbb{P}_{\mathcal{N}(0,\varepsilon)}$)

Exercise 3 — *Counterexample.*
Let $U = \{x \in \mathbb{R}^2, 0 < |x| < 1\} \subset \mathbb{R}^2$ be the punctured unit disk and let $\varphi : \partial U \to \mathbb{R}$ be the function defined by $\varphi(x) = \mathbf{1}_{x \neq 0}$. Consider the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = \varphi & \text{on } \partial U. \end{cases}$$

(1) Show that the Brownian expectation does not define a continuous solution of the equation above.

(2) Show that in fact this Laplace equation doesn't have any solution. (*Hint:* any continuous solution is of the form $u(x) = g(|x|)$)

Exercise 4 — *Gambler's ruin in several dimensions.*
Let $r, R \in (0, \infty)$ such that $r < R$ and $d \geq 1$, consider the annulus

$$U = \{x \in \mathbb{R}^d, r < |x| < R\}$$

Let B be a $\mathbb{R}^d$-valued Brownian motion, we let T_r (resp. T_R) denote the hitting time of the ball of radius r (resp. R) centered at 0 by B. The hitting time of ∂U by B is given by $T_{\partial U} = T_r \wedge T_R$. Recall the definition of the Laplacian operator,

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial^2 x_i}$$

(1) Let $c, C \in \mathbb{R}$ and let $\varphi : \partial U \to \mathbb{R}$ be the function defined by $\varphi(x) = c\mathbf{1}_{|x|=r} + C\mathbf{1}_{|x|=R}$. Give a solution of the Laplace equation

$$\begin{cases} \Delta u = 0 \text{ on } U \\ u = \varphi \text{ on } \partial U. \end{cases}$$

(2) Using u compute $\mathbb{P}_x(T_r < T_R)$.
(3) for $|x| > r$, compute $\mathbb{P}(T_r < \infty)$.
(4) Assume that $d = 2$, Show that almost surely, for every $x \in \mathbb{R}^2$ and $\varepsilon > 0$ there exists an increasing sequence $(t_n)_n \in (\mathbb{R}_+)^{\mathbb{N}}$ such that $t_n \to \infty$ and $|B(t_n) - x| \leq \varepsilon$.
(5) Assume that $d \geq 3$, show that $\mathbb{P}_0$-almost surely $\lim_{t\to\infty} |B_t| = \infty$. (*Hint:* Consider the events $A_n = \{$ for every $t > T_{n^3}$, $|B_t| > n\}$.)

Exercise 5 — *Law of iterated logarithms for random walks.*
Let B be a Brownian motion, define $\psi(t) = \sqrt{2t \log \log t}$.

(1) Let $(T_n)_n$ be a sequence of stopping times such that $T_n \to \infty$ almost surely and $T_n/T_{n+1} \to 1$ almost surely. For every $q > 4$, we define

$$D_k = \left\{B(q^k) - B(q^{k-1}) \geq \psi(q^k - q^{k-1})\right\}$$
$$\Omega_k = \left\{\min_{q^k \leq t \leq q^{k+1}} B(t) - B(q^k) - \sqrt{q^k}\right\}$$

In what follows we admit the existence of $c > 0$ such that $\mathbb{P}(D_k) \geq c/(k \log k)$.

(a) Show that $\limsup \frac{B(T_n)}{\psi(T_n)} \leq 1$ almost surely.
(b) Show that $\mathbb{P}(\limsup D_{2k} \cap \Omega_{2k}) = 1$.
(c) Show that almost surely for infinitely many $k \geq 1$, we have

$$\min_{q^k \leq t \leq q^{k+1}} B_t \geq \psi(q^k) \left(1 - \frac{1}{q} - \frac{2}{\sqrt{q}}\right) - \sqrt{q^k}$$

(*Hint:* You may use the inequality $\psi(q^k - q^{k-1}) \geq \psi(q^k)(1 - 1/q)$.)

(d) By considering the sequence $n(k) = \inf\{n \geq 1, T_n \geq q^k\}$, show that almost surely,

$$\limsup_{n\to\infty} \frac{B(T_n)}{\psi(T_n)} = 1.$$

(2) Define a sequence of stopping times recursively by $T_0 = 0$ and

$$T_{n+1} = \inf\{t > T_n, |B(t) - B(T_n)| = 1\},$$

show that $T_n/n \to 1$ almost surely.

(3) Let $(X_k)_k$ be a sequence of independent uniform random variables in $\{-1, 1\}$ and $S_n = \sum_{k=1}^n X_k$. Show that,

$$\limsup_{n\to+\infty} \frac{S_n}{\psi(n)} = 1.$$

Exercise 6 — *Quadratic and absolute variation.*
Let $t \geq 0$, a partition $\underline{t}$ of $[0, t]$ is a finite sequence $0 = t_0 \leq t_1 \leq \ldots \leq t_n = t$, given a partition we define its length $\#\underline{t} = n$ and its mesh-size $|\underline{t}| = \max_{1\leq i\leq \#\underline{t}} |t_i - t_{i-1}|$. Let $f : [0, t] \to \mathbb{R}$ be a measurable function, we define the total variation of f on $[0, t]$ by

$$TV_t(f) = \lim_{\epsilon\to 0} \sup_{|\underline{t}|\leq\epsilon} \sum_{i=1}^{\#\underline{t}} |f(t_i) - f(t_{i-1})|.$$

Where $\sup_{|\underline{t}|\leq\epsilon}$ should be understood as the supremum over all partitions of $[0, t]$ with mesh-size $\leq \epsilon$. Similarly, we define the quadratic variation of f on $[0, t]$ by

$$QV_t(f) = (\lim_{\epsilon\to 0} \sup_{|\underline{t}|\leq\epsilon} \sum_{i=1}^{\#\underline{t}} (f(t_i) - f(t_{i-1}))^2.$$

(1) Let $(\underline{t}^{(k)})_k$ be a sequence of partitions with $|\underline{t}^{(k)}| \to 0$. For every $k \geq 1$, let

$$X_k = \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2.$$

(a) Assume that the sequence $(X_k)_k$ converges in $L^2(\Omega)$ to some constant random variable X, show that $X = t$ almost surely.
(b) Show that $(X_k)_k$ converges in $L^2(\Omega)$ toward the constant random variable taking only the value t.
(c) Show that if $(\underline{t}^{(k)})_k$ is such that $\sum_{k=1}^{\infty} \sum_{j=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2 < \infty$, then $(X_k)_k$ converges almost surely.
(d) What can you say about the random variable $QV_t(B)$?

(2) Show that almost surely the trajectories of the Brownian do not have bounded total variation, that is $\mathbb{P}(TV_t(B) = \infty) = 1$. (*Hint*: what can you say about the quadratic variation of a continuous function with finite total variation ?).

Exercise 7 — *A weaker condition for the first Wald's lemma.*
We wish to show that when T is a stopping time with $\mathbb{E}[T^{1/2}] < \infty$, Wald's lemma still applies and $\mathbb{E}[B_T] = 0$

(1) Define $\tau := \min\{k : 4^k \geq T\}$. Set $M(t) := \max_{[0,t]} B$ and $X_k := M(4^k) - 2^{k+2}$. Show that (X_k) is a supermartingale for the filtration $(\mathcal{F}_{4^k})_k$, and that τ is a stopping time.
(2) Show that $\mathbb{E}[M(4^\tau)] < \infty$ and conclude.
(3) Show that when T is the hitting time of 1, we have $\mathbb{E}[T^\alpha] < \infty$ for all $\alpha < 1/2$ but $\mathbb{E}[B_T] \neq 0$. This proves that the exponent $1/2$ is optimal.