TD11 : Donsker's invariance principle and Arcsine Laws

Exercise 1 — *Another arcsine law.*
Let B be a Brownian motion on $[0,1]$.

(1) Let $[a_1, b_1]$ and $[a_2, b_2]$ be two non overlapping intervals ($b_1 \leq a_2$). Show that almost surely the maximum value of B on $[a_1, b_1]$ and $[a_2, b_2]$ are different.
(2) Using the previous question, show the following,
  (a) The global maximum of B on $[0,1]$ is attained at a unique point $M \in [0,1]$.
  (b) Every local maximum of B is a strict local maximum.
  (c) The set of points where the local maxima are attained is dense and countable.
(3) Show that for every $s \in [0,1]$, $\mathbb{P}(M \leq s) = \frac{2}{\pi} \arcsin(\sqrt{s})$.

Exercise 2 — *Yet another arcsine law.*
Let $(X_k)_{k\geq 1}$ be a sequence of iid standard random variables, let $(S_n)_{n\geq 0}$ be the random walk associated to $(X_k)_{k\geq 1}$. Let

$$N_n = \max\{k \in \{1, \dots, n\}, S_k S_{k-1} \leq 0\}$$

be the last sign change of (S_k) before time n. Given $f \in \mathcal{C}([0,1])$, let

$$G(f) = \sup\{t \in [0,1], f(t) = 0\}$$

denote its last zero. Let U denote the set of functions $f \in \mathcal{C}([0,1])$ such that $f(1) \neq 0$ and for every $t \in [0,1]$, if $f(t) = 0$ then for every $\varepsilon > 0$, the function f takes positive and negative values in $[t - \varepsilon, t + \varepsilon]$.

(1) Recall how to define $S_n^* \in \mathcal{C}([0,1])$ using the trajectories of the random walk $(S_n)_n$.
(2) Let B be a Brownian motion, show that $G(B)$ is arcsine distributed. (*Hint*: show that $G(B) \overset{(d)}{=} M$ where M is the random variable of exercise 1.)
(3) Show that for every $f \in U$, the function G is continuous at f.
(4) Show that almost surely G is continuous at B. (*Hint: use Exercise 1 of TD11 and Exercise 4 of TD9*)
(5) Show that N_n/n converges in law to N, where $\mathbb{P}(N \leq x) = \frac{2}{\pi}\arcsin(\sqrt{x})$.

Exercise 3 — *Maximum value of a random walk.*
Let $(X_k)_{k\geq 1}$ be a sequence of iid standard random variables, let $(S_n)_{n\geq 0}$ be the random walk associated to $(X_k)_{k\geq 1}$. Define,

$$M_N = \sup\{S_n, 0 \leq n \leq N\}.$$

Compute the limit in law of $M_N/\sqrt{N}$ as $N \to \infty$.