TD9 : Regularity of Brownian Trajectories and Stopping Times

Exercise 1 — *Warm-up.*
Let B be a Brownian motion, using the fact that B is a Gaussian process, show that B has the simple Markov property. That is for every $s \geq 0$ show that $(B_{t+s} - B_s)_{t\geq 0}$ is a Brownian motion independent from $(B_t)_{t\leq s}$.

Exercise 2 — *Counter-example.*
Let B be a Brownian motion and let

$$T = \inf\{t \geq 0, B_t = \max_{s\in[0,1]} B_s\},$$

be the first hitting time by B of the maximum of B on $[0,1]$, is T a stopping time ?

Exercise 3 — *Hölder regularity of Brownian trajectories.*
Let B be a Brownian motion, recall from Exercise 4 of TD8 that the process $X = (tB_{1/t})_{t\geq 0}$ is also a Brownian motion.

(1) Show that $\lim_{t\to\infty} \frac{B_t}{t} = 0$ a.s. and that for every $\varepsilon > 0$,

$$\lim_{t\to 0} \frac{B_t}{t^{1/2-\varepsilon}} = 0 \text{ and } \lim_{t\to\infty} \frac{B_t}{t^{1/2+\varepsilon}} = 0 \text{ a.s..}$$

(2) Let (ξ_n) be a sequence of independent and identically distributed centered random variables with variance 1,
  (a) For every $K \in \mathbb{R}$, show that $\mathbb{P}(\limsup_{n\to\infty}\{\sum_{k=1}^n \xi_k \geq K\sqrt{n}\}) > 0$.
  (b) Show that almost surely

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^n \xi_k = +\infty.$$

(3) Deduce from the previous question that,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty,$$

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t\to 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

(4) Recall that given a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $t \geq 0$, we say that f is locally α-Hölder at t if there exists a neighborhood V of t in $\mathbb{R}_+$ and a constant $c \geq 0$ such that for every $s, s' \in V$, we have

$$|f(s) - f(s')| \leq c|s - s'|^\alpha.$$

  (a) Show that almost surely B is not locally 1/2-Hölder at 0.

(b) For every $t \geq 0$, define $L_t = \limsup_{s\to 0} \left|\frac{B_{t+s}-B_t}{\sqrt{s}}\right|$ and $A = \{t \in \mathbb{R}_+, L_t < +\infty\}$. Show that almost surely the set A is negligible with respect to the Lebesgue measure.

(c) Show that the set of points at which B is locally 1/2-Hölder is almost surely negligible with respect to the Lesbesgue measure.

Exercise 4 — *The set of zeros of B is perfect.*
Recall that for every $S \subset \mathbb{R}$ we say that $x \in S$ is an isolated point of S if there exists a neighborhood V of x in $\mathbb{R}$ such that $V \cap S = \{x\}$. Let B be a Brownian motion, and

$$Z = \{t \geq 0, \ B_t = 0\}.$$

(1) Show that the following events are almost sure,
   (a) Z is infinite and is a closed set.
   (b) Z has Lebesgue measure 0.
   (c) Z has no isolated points.
(2) ($\star$) Make the notion of random closed set rigorous by defining a σ-algebra on the set of closed subsets of $\mathbb{R}_+$.

Exercise 5 — *Another counter-example.*
Let $X_t = AB_t$ where B is a Brownian motion started from 1 and A an independent balanced Bernoulli.

(1) Show that X is a Markov process and give its transition kernel.
(2) Show that it does not verify the strong Markov property.

Exercise 6 — *Brownian motion on the circle.*
Define a Brownian motion on the circle $\mathbb{S}^1$ by setting $X_t = e^{iB_t}$ for $t \geq 0$. What is the distribution of the last point hit by X in $\mathbb{S}^1$?