TD8 : Construction of the Brownian Motion

Exercise 1 — *Transformations.*
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(1) Show that for any $\lambda \in \mathbb{R}_+^\star$, the process $(\lambda^{-1/2} B_{\lambda t})_{t\geq 0}$ is a Brownian motion.
(2) Show that $B_1 - B_{1-t}$ is a Brownian motion on $[0,1]$.

Exercise 2 — *Constructing a Brownian motion indexed by $\mathbb{R}_+$.*
Let $(B^{(n)})_n$ be a sequence of independent Brownian motions defined on $[0,1]$. For every $t \geq 0$, define

$$B_t = B^{(\lfloor t \rfloor)}_{t-\lfloor t \rfloor} + \sum_{i=0}^{\lfloor t \rfloor - 1} B_1^{(i)}.$$

Show that $(B_t)_{t\geq 0}$ is a Brownian motion.

Exercise 3 — *Lévy's construction of the Brownian motion.*
Let $H = L^2([0,1])$ with the usual inner product. For $t \geq 0$ let $I_t = \mathbb{1}_{[0,t]} \in H$. We also set $(e_i)_{i\in\mathbb{N}}$ to be an orthonormal basis of H.

(1) Check that $\langle I_s, I_t \rangle = s \wedge t$.
(2) Assume that there exists a H-valued standard Gaussian random variable. That is, a random variable $\xi \in H$, such that for every $x \in H$, $\langle x, \xi \rangle \sim \mathcal{N}(0, |x|^2)$.
  (a) Using the random variable ξ and the functions $(I_t)_{t\geq 0}$, build a Gaussian process $(B_t)_{t\in[0,1]}$ such that $\mathrm{Cov}(B_s, B_t) = s \wedge t$.
  (b) Let $Z_i = \langle \xi, e_i \rangle$, so that $\xi = \sum_{i\in\mathbb{N}} Z_i e_i$. Show that the (Z_i) are independent standard Gaussians (*Hint:* Compute the characteristic function of finite subvectors.). Deduce that the process of the previous question would satisfy,

$$B_t = \sum_{n=0}^{\infty} Z_n \int_0^t e_i(s) ds. \qquad (\dagger)$$

  (c) By computing $|\xi|^2$, show that ξ cannot exist.
(3) Define $h_0 = 0$ and for $n \geq 0$ and $0 \leq k < 2^n$,

$$h_{k,n} := 2^{n/2} \left(\mathbb{1}_{\left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right]} - \mathbb{1}_{\left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right]} \right),$$

We admit (or recall) that $(h_{k,n})_{k,n}$ is an orthonormal basis of H called the Haar wavelet basis. Let $(Z_{n,k})_{n,k}$ be a family of independent standard Gaussian random variables. For every $t \geq 0$ set

$$B_t = tZ + \sum_{n=0}^{\infty} F_n(t), \tag{††}$$

where $F_n(t) = \sum_{k=0}^{2^n-1} Z_{n,k} f_{n,k}(t)$ and $f_{n,k}(t) = \int_0^t h_{n,k}(s)ds$.

(a) Using the inequality $\mathbb{P}(|Y| \geq \lambda) \leq \frac{\sqrt{2/\pi}}{\lambda} e^{-\lambda^2}$ for $\lambda > 0$ and $Y \sim \mathcal{N}(0,1)$, show that

$$\mathbb{P}\left(2^{-\frac{n+2}{2}} \max_{0 \leq k < 2^n} |Z_{n,k}| > \frac{1}{n^2}\right) = o\left(\frac{1}{n^2}\right).$$

(b) Show that $\mathbb{P}\left(\|F_n\|_\infty \leq \frac{1}{n^2} \text{ for } n \text{ large enough }\right) = 1$.

(c) Show that almost surely, the sum of functions in (††) converges uniformly on $[0,1]$ to a (random) continuous function.

(4) ($\star$) Prove the same result than in the previous question when we use the Fourier basis $e_0 = 1$, and $e_m(t) = \sqrt{2}\cos(\pi m t)$ in (†) rather than the Haar wavelet basis.

Exercise 4 — *Time inversion.*

Let $(B_t)_{t \geq 0}$ be a Brownian motion. Set $X_t = tB_{1/t}$ for $t > 0$ and $X_0 = 0$.

(1) Show that X has the finite-dimensional marginals of a Brownian motion.

(2) Show that the set $U = \{f \in \mathbb{R}^{\mathbb{Q}_+}, \lim_{t \to 0, t \in \mathbb{Q}} f_t = 0\} \subset \mathbb{R}^{\mathbb{Q}_+}$ is measurable.

(3) Deduce that $(X_t)_t$ is continuous almost surely, hence may be modified on a negligible event to form a Brownian motion.