TD3 : Continuous time Markov chains

Exercice 1 — *Transition semigroups.*
In this exercise, questions (3) is independent from the rest.

(1) Let $d \geq 1$, let $(\mu_t)_{t\geq}$ be a family of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Assume that there exists a measurable function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that for every $t \geq 0$, the characteristic function of μ_t is given by $\xi \mapsto e^{t\phi(\xi)}$. Let $(X_t)_{t\geq 0}$ be independent random variables such that for every $t \geq 0$, X_t has law μ_t. For every $t \geq 0$ and $x \in \mathbb{R}^d$, let $P_t(x, \cdot)$ be the law of $x + X_t$. Show that $(P_t)_{t\geq 0}$ is a Markov semigroup on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

(2) Using the previous question, show that the Poisson semigroup, the Gaussian semigroup, and the Cauchy semigroup from lecture 2 are indeed Markov semigroups. Briefly explain why it's not possible to use question (1) to show that the lognormal semigroup is indeed a Markov semigroup.

(3) Let $(E, \mathcal{E})$ be a measurable space, let $(P_t)_{t\geq 0}$ be a Markov semigroup on $(E, \mathcal{E})$ and let $h : \times E \to \mathbb{R}_+^*$ be a measurable function. For every $t \geq 0$ and every $x \in \mathbb{R}^d$, we define a measure $\hat{P}_t(x, \cdot)$ on E via,
$$\frac{d\hat{P}_t(x, \cdot)}{dP_t(x, \cdot)} = \frac{1}{h(x)} h(\cdot).$$
Assume that for every $t \geq 0$ and every $x \in \mathbb{R}^d$, we have,
$$h(x) = \int_E h(x') dP_t(x, dx').$$
Show that $(\hat{P}_t)_{t\geq 0}$ is a Markov semigroup on $(E, \mathcal{E})$.

Exercice 2 — *Thinning property.*
Let $(N_t)_{t\geq 0}$ be a Poisson process of intensity $\lambda > 0$ and $(X_k)_{k\geq 0}$ be iid Bernoulli random variables with parameter $p \in [0, 1]$. Let
$$N_t^A = \sum_{k=1}^{N_t} X_k \text{ and } N_t^B = \sum_{k=1}^{N_t} (1 - X_k),$$

(1) Let X a Poisson random variable of parameter $\alpha > 0$ and $(B_n)_n$ be iid Bernoulli random variables of parameter p independent from X, show that $Y = \sum_{n=1}^{X} B_n$ is a Poisson random variable of parameter αp.

(2) Show that N^A and N^B are independent Poisson processes with intensity λp and $\lambda(1 - p)$.

(3) A bus station observes arrivals of buses, modeled as a Poisson process of intensity λ. Each arriving bus is either a city bus with probability p, or an intercity bus with

probability $1-p$ independently of other buses. What is the law of the time between each city bus ? each intercity bus ? Given $t \geq 0$ what is the expected number of city buses observed at time t given that we have observed $n \in \mathbb{N}$ buses in total ?

Exercice 3 — *$M/GI/\infty$ queue.*

Let $X = (X_t)_{t\geq 0}$ be a Poisson process of intensity $\lambda > 0$, we denote $(J_n)_n$ the jump times of X. Let $(Z_n)_n$ be iid random variables, we denote G the cdf of Z_1 and $1/\mu$ the mean of Z_1. Consider the following model, you operate a restaurant in which the n^{th} customer arrives at time J_n and leaves at time $J_n + Z_n$. You want to estimate the number N_t of customers in the shop at time t. Note that for every $t \geq 0$, we have

$$N_t = \sum_n \mathbf{1}\{J_n \leq t \leq J_n + Z_n\}.$$

(1) Let $t \geq 0$, $n \geq 0$ and let U denote a uniform random variable in $[0, t]$, define $p = \mathbb{P}(Z_1 > U)$. Show that conditionally on $X_t = n$, the random variable N_t is Binomial random variable with parameter (n, p).
(2) Let $t > 0$ and $\alpha(t) = \lambda \int_0^t \mathbb{P}(Z_1 > x)dx$, show that N_t is a Poisson random variable with parameter $\alpha(t)$.
(3) Show that as $t \to \infty$, N_t converges in law toward a Poisson law with parameter $\rho = \lambda/\mu$.

In France approximately, 1903896 new cars have been bought each year between 1967 and 2023 (source : CCFA, Comité des Constructeurs Français d'Automobiles). Assume that the French people buy cars according to a Poisson Process with intensity $\lambda = 1903896$ per year and that there was no car bought before 1967.

(4) Assume that each car owner keeps its car for a duration uniform between 0 and 20 years. What is the expected number of cars in the French fleet in the year 1977 ? what about in the year 1987 ? and Afterward ?
(5) Answer the previous question now assuming that each owner keeps its car for an exponential duration with parameter $1/10$.