ENS de Lyon — Math Department Master 1 — Spring 2025
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TD12 : Harmonic Functions and Miscellaneous

Exercise 1 — *Area of planar Brownian motion.*
Let B be a planar Brownian motion. let $\mathcal{L}^2$ denote the Lebesgue measure on $\mathbb{R}^2$. We are going to show that almost surely $\{B_t, t \in [0,1]\}$ is $\mathcal{L}^2$-negligible.

(1) Let $A_1, A_2 \subset \mathbb{R}^2$ be measurable sets, show that if $\mathcal{L}^2(A_1), \mathcal{L}^2(A_2) > 0$ then

$$\mathcal{L}^2\left\{x \in \mathbb{R}^2 : \mathcal{L}^2(A_1 \cap (x + A_2)) > 0\right\} > 0.$$

If we prove the result for A_1, A_2 bounded sets, then the result follows since

$$\mathcal{L}^2\left\{x \in \mathbb{R}^2 : \mathcal{L}^2(A_1 \cap (x + A_2)) > 0\right\} \geq \mathcal{L}^2\left\{x \in \mathbb{R}^2 : \mathcal{L}^2(A_1^M \cap (x + A_2^M)) > 0\right\},$$

where $A_i^M = B(0, M) \cap A_i$. In what follows we thus assume that A_1 and A_2 are bounded sets. We have

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathcal{L}^2(A_1 \cap (x + A_2))\mathrm{d}x &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}_{A_1}(y) 1_{x+A_2}(y)\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}_{A_1}(y) 1_{-A_2}(x-y)\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}_{A_1}(y) 1_{-A_2}(x-y)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2} {}_{A_1}(y) \left(\int_{\mathbb{R}^2} {}_{A_1}(y) 1_{-A_2}(x-y)\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{\mathbb{R}^2} {}_{A_1}(y) \mathcal{L}^2(A_2) \\
&= \mathcal{L}^2(A_1)\mathcal{L}^2(A_2).
\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^2} \mathcal{L}^2(A_1 \cap (x + A_2))\mathrm{d}x = \mathcal{L}^2(A_1)\mathcal{L}^2(A_2) > 0.$$

By contradiction, if we have $\mathcal{L}^2\{x \in \mathbb{R}^2 : \mathcal{L}^2(A_1 \cap (x + A_2)) > 0\} = 0$, then

$$\int_{\mathbb{R}^2} \mathcal{L}^2(A_1 \cap (x + A_2))\mathrm{d}x = 0,$$

a contradiction!

(2) Let $X_I = \mathcal{L}^2\{B_t, t \in I\}$, by bounding $\mathbb{P}(X_I > a)$, show that for every bounded interval $I \subset \mathbb{R}_+$, the expectation of X_I is finite.

Since I is bounded it is contained in an interval of the form $[0,T]$. When $X_{[0,T]} > a$, the set $\{B_t, t \leq T\}$ is not contained in $[-\sqrt{a}/2, \sqrt{a}/2]$. Thus, if W denote a 1-dimensionnal Brownian motion we have

$$\mathbb{P}(X_I > a) \leq 2\mathbb{P}(\max_{t\leq T} W_t > a) = 4\mathbb{P}(W_T > a) \leq c_1 e^{-c_2 a}.$$

Thus $a \mapsto \mathbb{P}(X_I > a)$ is integrable on $\mathbb{R}_+$ and $\mathbb{E}\, X_I < +\infty$.

(3) Show that $\mathcal{L}^2\{B_t, t \in [0,1]\} \cap \mathcal{L}^2\{B_t, t \in [2,3]\} = 0$ almsot surely.

Recall that $(B_{3t})_{t\geq 0}$ and $(\sqrt{3}B_t)_{t\geq 0}$ have the same law. Hence, $\mathbb{E}\, X_{[0,3]} = 3\,\mathbb{E}\, X_{[0,1]}$. In addition, since X_I defines a nonnegative random variable, by additivity of the Lebesgue measure we have,

$$\mathbb{E}\, X_{[0,3]} \leq \mathbb{E}\, X_{[0,1]} + \mathbb{E}\, X_{[1,2]} + \mathbb{E}\, X_{[2,3]},$$

with equality if and only if $\mathbb{E}[\mathcal{L}^2\{B_t, t \in [i, i+1]\} \cap \mathcal{L}^2\{B_t, t \in [j, j+1]\}] = 0$ for $i \neq j$. On the other hand, the inequality in the previous display implies that

$$3\,\mathbb{E}\, X_{[0,1]} = \mathbb{E}\, X_{[0,3]} \leq \mathbb{E}\, X_{[0,1]} + \mathbb{E}\, X_{[1,2]} + \mathbb{E}\, X_{[2,3]} \leq 3\,\mathbb{E}\, X_{[0,1]}.$$

Thus, we have $\mathbb{E}[\mathcal{L}^2\{B_t, t \in [i, i+1]\} \cap \mathcal{L}^2\{B_t, t \in [j, j+1]\}] = 0$ for $i \neq j$

(4) Let $B'_t = B_{t+2} - B_2 + B_1$ and

$$R(x) = \mathcal{L}^2\{B_t : t \in [0,1]\} \cap \{x + B'_t : t \in [0,1]\}.$$

Show that for almost all $x \in \mathbb{R}^2$, we have $\mathbb{P}(R(x) = 0) = 1$.

Let $Y = B_2 - B_1$, by the Markov property Y is independent of B and B' and is a standard normal. It follows from (1) that

$$0 = \mathbb{E}\left[\mathcal{L}^2\{B_t, t \in [0,1]\} \cap \mathcal{L}^2\{B_t, t \in [2,3]\}\right] = \mathbb{E}\, R[Y].$$

Thus for almost all $x \in \mathbb{R}^2$, $R(x) = 0$ almost surely.

(5) Deduce from the previous question that almost surely,

$$\mathcal{L}^2\{x \in \mathbb{R}^2 : R(x) > 0\} = 0.$$

This is a standard application of the Fubini lemmma,

$$\begin{aligned}
\mathbb{E}\mathcal{L}^2\{x \in \mathbb{R}^2 : R(x) > 0\} &= \mathbb{E}\int_{\mathbb{R}_2} \mathbf{1}_{R(x)>0}\mathrm{d}x \\
&= \int_{\mathbb{R}_2} \mathbb{E}\,\mathbf{1}_{R(x)>0}\mathrm{d}x \\
&= \int_{\mathbb{R}_2} \mathbb{P}(R(x) > 0)\mathrm{d}x \\
&= 0.
\end{aligned}$$

Thus the result.

(6) Deduce from the previous questions that almost surely $\{B_t : t \in [0,1]\}$ is $\mathcal{L}^2$-negligible. From question (1) and the preivous question, we have almost surely $X_[0,1] = 0$ or $X_[2,3] = 0$. Since $X_[0,1]$ and $X_[2,3]$ are independent and follow the same law we obtain that $X_[0,1] = 0$ almosy surely, this is the desired result.

Exercise 2 — *Point transience of Brownian motion.*
Let $d \geq 2$ and B a Brownian motion on $\mathbb{R}^d$. We wish to show that for every $y \in \mathbb{R}^d$, we have

$$\mathbb{P}(y \in \{B_t, t \in (0,1]\}) = 0.$$

(1) Show that the result for $d = 2$ implies the result for all $d \geq 2$. In the rest of the exercise we assume that $d = 2$.

We can project on the first two coordinates. Indeed, If we write $B_t = (B_t^i)_{1 \leq i \leq d}$, then $B'_t = (B_t^1, B_t^2)$ is a Brownian motion in $\mathbb{R}^2$. We can the observe that for $y = (y_i)_{1 \leq i \leq d} \in \mathbb{R}^d$,

$$\mathbb{P}(y \in \{B_t, t \in (0,1]\}) \leq \mathbb{P}((y_1, y_2) \in \{B'_t, t \in (0,1]\})$$

and that the probability on the right hand side is 0 when we know that the result of the exercise for $d = 2$.

(2) Show for every $x \in \mathbb{R}^d$ and almost all $y \in \mathbb{R}^d$, $\mathbb{P}_x(y \in \{B_t, t \in (0,1]\}) = 0$.

Since $d = 2$, we can use the main result of the previous exercise: $\mathcal{L}^2\{B_t,\, t \in [0,1]\} = 0$ almsot surely. This is independent of the starting of the starting point so forall $x \in \mathbb{R}^2$ we have $\mathcal{L}^2\{B_t,\, t \in [0,1]\} = 0$ $\mathbb{P}_x$-almost surely. By applying Fubini's theorem, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathbb{P}_{\frown}(y \in \{B_t,\, t \in [0,1]\})\mathrm{d}y &= \int_{\mathbb{R}^2} \mathbb{E}_x\, \mathbf{1}_{y \in \{B_t,\, t \in [0,1]\}} \mathrm{d}y \\
&= \mathbb{E}_x \int_{\mathbb{R}^2} \mathbf{1}_{y \in \{B_t,\, t \in [0,1]\}} \mathrm{d}y \\
&= \mathbb{E}_x\, \mathcal{L}^2\{B_t,\, t \in [0,1]\} \\
&= 0.
\end{aligned}$$

(3) Deduce that $y \in \mathbb{R}^2$ and almost all $x \in \mathbb{R}^2$, $\mathbb{P}_x(y \in \{B_t, t \in (0,1]\}) = 0$.

By symmetry of the Brownian motion, we can inverse the role of x and y. Let $y \in \mathbb{R}^2$, for almost all $x \in \mathbb{R}^2$ it holds that

$$\begin{aligned}
\mathbb{P}_x(y \in \{B_t, t \in (0,1]\}) &= \mathbb{P}_0(y \in \{x + B_t, t \in (0,1]\}) \\
&= \mathbb{P}_0(y - x \in \{B_t, t \in (0,1]\}) \\
&= \mathbb{P}_y(x \in \{B_t, t \in (0,1]\}) \\
&= 0.
\end{aligned}$$

(4) Conclude. (Hint: you may consider $\mathbb{P}_{\mathcal{N}(0,\varepsilon)}$)

In light of the previous question, we have for all $y \in \mathbb{R}^2$, $\mathbb{P}_{\mathcal{N}(0,\varepsilon)}(y \in \{B_t, t \in (0,1]\}) = 0$. But then,

$$\begin{aligned}\mathbb{P}_0(y \in \{B_t, t \in (0,1]\}) &= \lim_{\varepsilon\to 0} \mathbb{P}_x(y \in \{B_t, t \in (\varepsilon, 1]\}) \\ &= \lim_{\varepsilon\to 0} \mathbb{E}_x\, \mathbb{P}_{\mathcal{N}(0,\varepsilon)}(y \in \{B_t, t \in (0, 1-\varepsilon]\}) \\ &= 0.\end{aligned}$$

Exercise 3 — *Counterexample.*
Let $U = \{x \in \mathbb{R}^2, 0 < |x| < 1\} \subset \mathbb{R}^2$ be the punctured unit disk and let $\varphi : \partial U \to \mathbb{R}$ be the function defined by $\varphi(x) = \mathbf{1}_{x\neq 0}$. Consider the Laplace equation

$$\begin{cases}\Delta u = 0 & \text{on } U \\ u = \varphi & \text{on } \partial U.\end{cases}$$

(1) Show that the Brownian expectation does not define a continuous solution.
Set $T = T_{\partial U}$ be the hitting time of the boundary of U and $u(x) = \mathbb{E}_x[\varphi(B_T)1_{T<\infty}]$. This does not define a continuous solution to the Laplace equation, because since the Brownian motion started outside of 0 almost surely does not hit 0, we have $u(0) = 0$ and $h(x) = 1$ for all $x \in \overline{U} \setminus \{0\}$. Hence h is not continuous.

(2) Show that in fact this Laplace equation doesn't have any solution. (*Hint:* any continuous solution is of the form $u(x) = g(|x|)$) By contradiction, let u be a continuous solution. For every $\theta \in [0, 2\pi)$, let R_θ denote the planar rotation of angle θ, we start by showing that $u \circ R_\theta = u$. The function $v = u \circ R_\theta$ is continuous, and we have $\Delta v = \Delta u = 0$. In addition, on ∂U, $v = \varphi \circ R_\theta = \varphi$. By uniqueness, it follows that $v = u$. In particular since u is invariant by roation, there exists $g : [0,1] \to \mathbb{R}$ such that $u(x) = g(|x|)$. We have,

$$\begin{cases} g''(x) + \frac{1}{x} g'(x) = 0 \text{ on } (0,1) \\ (g(0), g(1)) = (0,1).\end{cases}$$

The solutions of this ODE are of the form $x \mapsto A + B\log(x)$, and we must have $B = 0$ and $A = 0$ to satisfy the condition, $g(0) = 0$ but then $g(1) = 0$, a contradiction.

Exercise 4 — *Gambler's ruin in several dimensions.*
Let $r, R \in (0,\infty)$ such that $r < R$ and $d \geq 1$, consider the annulus

$$U = \{x \in \mathbb{R}^d, r < |x| < R\}$$

Let B be a $\mathbb{R}^d$-valued Brownian motion, we let T_r (resp. T_R) denote the hitting time of the ball of radius r (resp. R) centered at 0 by B. The hitting time of ∂U by B is given by

$T_{\partial U} = T_r \wedge T_R$. Recall the definition of the Laplacian operator,

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial^2 x_i}$$

(1) Let $c, C \in \mathbb{R}$ and let $\varphi : \partial U \to \mathbb{R}$ be the function defined by $\varphi(x) = c\mathbf{1}_{|x|=r} + C\mathbf{1}_{|x|=R}$. Give a solution of the Laplace equation

$$\begin{cases} \Delta u = 0 \text{ on } U \\ u = \varphi \text{ on } \partial U. \end{cases}$$

The problem is radially symmetric, so we look for solutions in the set of radially symmetric functions $u(x) = v(|x|^2)$, the condition u imposes,

$$\begin{cases} v''(y) + \frac{d-1}{y} v'(y) = 0 \text{ on } (r, R) \\ (v(r), v(R)) = (c, C) \end{cases}$$

Solutions of this equation satisfy $v'(x) = Ay^{1-d}$, integrating we discover that up to an additive and a multiplcative constant, we have,

$$u(x) = \begin{cases} |x|^{2-d} \text{ if } d \neq 2 \\ \log |x| \text{ otherwise} \end{cases}$$

(2) Using u compute $\mathbb{P}_x(T_r < T_R)$. Let $p_x = \mathbb{P}_x(T_r < T_R)$. For every $x \in \overline{U}$, $u(x) = \mathbb{E}_x[u(B_T)\mathbf{1}_{T<\infty}] = \mathbb{E}_x[u(B(T_r))\mathbf{1}_{T_r<T_R}] + \mathbb{E}_x[u(B(T_R))\mathbf{1}_{T_R<T_r}] = cp_x + C(1 - p_x)$. We have,

$$p_x = \frac{C - u(x)}{C - c}$$

Plugging in the values we have found for u we discover,

$$\mathbb{P}_x(T_r < T_R) = \begin{cases} \frac{R-|x|}{R-r} \text{ when } d = 1 \\ \frac{\log R - \log |x|}{\log R - \log r} \text{ when } d = 2 \\ \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}} \text{ when } d \geq 3. \end{cases}$$

(3) for $|x| > r$, compute $\mathbb{P}(T_r < \infty)$.

Repeating the argument but with harmonic functions on the complement of the ball or radius r or simply letting $R \to \infty$ in the previous formula, we obtain

$$\mathbb{P}_x(T_r < T_R) = \begin{cases} 1 \text{ when } d \leq 2 \\ \left(\frac{r}{|x|}\right)^{d-2} \text{ when } d \geq 3. \end{cases}$$

(4) Assume that $d = 2$, Show that almost surely, for every $x \in \mathbb{R}^2$ and $\varepsilon > 0$ there exists an increasing sequence $(t_n)_n \in (\mathbb{R}_+)^{\mathbb{N}}$ such that $t_n \to \infty$ and $|B(t_n) - x| \leq \varepsilon$.

Fix $x \in \mathbb{Q}^2$ and $\varepsilon \in (0,\infty) \cap \mathbb{Q}$, we define a sequence of stopping times by,

$$\begin{cases} \tau_1 = \inf\{t \geq 0, |B_t - x| \leq \varepsilon\} \\ \tau_{n+1} = \inf\{t \geq \tau_n + 1, |B_t - x| \leq \varepsilon\} \end{cases}$$

By the previous question τ_1 is almost surely finite, by induction and applying the strong Markov property to $1+\tau_n$, we discover that for every n, τ_n is almost surely finite. We have built an increasing sequence $(\tau_n)_n$ such that almost surely $\tau_n \to \infty$ and $|B_{\tau_n} - x| \leq \varepsilon$. Therefore, almost surely, the following holds, for every $x \in \mathbb{Q}^2$ and $\varepsilon \in (0,\infty) \cap \mathbb{Q}$ there exists an increasing sequence $(t_n)_n \in (\mathbb{R}_+)^{\mathbb{N}}$ such that $t_n \to \infty$ and $|B(t_n) - x| \leq \varepsilon$. Finally, we can recover the result for $x \in \mathbb{R}^2$ and $\varepsilon > 0$ by density.

(5) Assume that $d \geq 3$, show that $\mathbb{P}_0$-almost surely $\lim_{t\to\infty} |B_t| = \infty$. (*Hint:* Consider the events $A_n = \{$ for every $t > T_{n^3}$, $|B_t| > n\}$.)

Consider the events,

$$A_n = \{ \text{ for every } t > T_{n^3},\ |B_t| > n\}.$$

$\mathbb{P}_0$-almost surely T_{n^3} is finite, so by Markov's property,

$$\mathbb{P}_0(A_n^c) = \mathbb{E}_0[\mathbb{P}_{B(T_{n^3})}(T_n < \infty))] = \left(\frac{1}{n^2}\right)^{d-2}.$$

By Borel-Cantelli, $\mathbb{P}_0(\limsup A_n) = 1$. That is, almost surely A_n is realised for infinitely n, but on this even $\lim_{t\to\infty} |B_t| = \infty$

Exercise 5 — *Law of iterated logarithms for random walks.*
Let B be a Brownian motion, define $\psi(t) = \sqrt{2t \log\log t}$.

(1) Let $(T_n)_n$ be a sequence of stopping times such that $T_n \to \infty$ almost surely and $T_n/T_{n+1} \to 1$ almost surely. For every $q > 4$, we define

$$D_k = \left\{B(q^k) - B(q^{k-1}) \geq \psi(q^k - q^{k-1})\right\}$$
$$\Omega_k = \left\{ \min_{q^k \leq t \leq q^{k+1}} B(t) - B(q^k) - \sqrt{q^k} \right\}$$

In what follows we admit the existence of $c > 0$ such that $\mathbb{P}(D_k) \geq c/(k \log k)$.

(a) Show that $\limsup \frac{B(T_n)}{\psi(T_n)} \leq 1$ almost surely.
(b) Show that $\mathbb{P}(\limsup D_{2k} \cap \Omega_{2k}) = 1$.
(c) Show that almost surely for infinitely many $k \geq 1$, we have

$$\min_{q^k \leq t \leq q^{k+1}} B_t \geq \psi(q^k)\left(1 - \frac{1}{q} - \frac{2}{\sqrt{q}}\right) - \sqrt{q^k}$$

(*Hint:* You may use the inequality $\psi(q^k - q^{k-1}) \geq \psi(q^k)(1 - 1/q)$.)

(d) By considering the sequence $n(k) = \inf\{n \geq 1, T_n \geq q^k\}$, show that almost surely,

$$\limsup_{n\to\infty} \frac{B(T_n)}{\psi(T_n)} = 1.$$

(2) Define a sequence of stopping times recursively by $T_0 = 0$ and

$$T_{n+1} = \inf\{t > T_n, |B(t) - B(T_n)| = 1\},$$

show that $T_n/n \to 1$ almost surely.

(3) Let $(X_k)_k$ be a sequence of independent uniform random variables in $\{-1, 1\}$ and $S_n = \sum_{k=1}^n X_k$. Show that,

$$\limsup_{n\to\infty} \frac{S_n}{\psi(n)} = 1.$$

You can find a detailled correction of this exercise (and much more!) by looking at the proof of Theorem 5.4 in the book Brownian Motion by P. Morters and Y. Peres which you can find here.

Exercise 6 — *Quadratic and absolute variation.*
Let $t \geq 0$, a partition $\underline{t}$ of $[0,t]$ is a finite sequence $0 = t_0 \leq t_1 \leq \ldots \leq t_n = t$, given a partition we define its length $\#\underline{t} = n$ and its mesh-size $|\underline{t}| = \max_{1\leq i\leq \#\underline{t}} |t_i - t_{i-1}|$. Let $f : [0,t] \to \mathbb{R}$ be a measurable function, we define the total variation of f on $[0,t]$ by

$$TV_t(f) = \lim_{\epsilon\to 0} \sup_{|\underline{t}|\leq\epsilon} \sum_{i=1}^{\#\underline{t}} |f(t_i) - f(t_{i-1})|.$$

Where $\sup_{|\underline{t}|\leq\epsilon}$ should be understood as the supremum over all partitions of $[0,t]$ with mesh-size $\leq \epsilon$. Similarly, we define the quadratic variation of f on $[0,t]$ by

$$QV_t(f) = (\lim_{\epsilon\to 0} \sup_{|\underline{t}|\leq\epsilon} \sum_{i=1}^{\#\underline{t}} (f(t_i) - f(t_{i-1}))^2.$$

(1) Let $(\underline{t}^{(k)})_k$ be a sequence of partitions with $|\underline{t}^{(k)}| \to 0$. For every $k \geq 1$, let

$$X_k = \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2.$$

(a) Assume that the sequence $(X_k)_k$ converges in $L^2(\Omega)$ to some constant random variable X, show that $X = t$ almost surely. Assume that $X_k \xrightarrow{L^2} X$, then $|\mathbb{E}\, X_k - \mathbb{E}\, X| \leq \mathbb{E}[|X_k - X|^2]^{1/2} \to 0$. We have,

$$\mathbb{E}\, X_k = \sum_{i=1}^{\#\underline{t}^{(k)}} \mathbb{E}(B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 = \sum_{i=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)}) = t.$$

Therefore $X = \mathbb{E}\, X = \lim \mathbb{E}\, X_k = t$.

(b) Show that $(X_k)_k$ converges in $L^2(\Omega)$ toward the constant random variable taking only the value t.
Let Z be a standard Gaussian random variable and let $c = \mathrm{Var}(Z^2 - 1)$. Let $A_k = X_k - t$, $(A_k)_k$ is a sequence of centered random variables. By independence of the increments of B we have,

$$\begin{aligned}\mathbb{E}\, A_k^2 &= \mathrm{Var}(A_k)\\ &= \sum_{i=1}^{\#\underline{t}^{(k)}} \mathrm{Var}\left((B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 - (t_i^{(k)} - t_{i-1}^{(k)})\right)\\ &= \sum_{i=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2 c\\ &\leq ct|\underline{t}^{(k)}|.\end{aligned}$$

So $\mathbb{E}\, A_k^2 \to 0$ and the result is proven.

(c) Show that if $(\underline{t}^{(k)})_k$ is such that $\sum_{k=1}^{\infty} \sum_{j=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2 < \infty$, then $(X_k)_k$ converges almost surely. According to the computation of the previous question $\mathbb{E}\, A_k^2 \leq c \sum_{i=1}^{\#\underline{t}^{(k)}} (t_i^{(k)} - t_{i-1}^{(k)})^2$, so by assumption $\sum_k \mathbb{E}\, A_k^2 < \infty$. For every $\varepsilon > 0$, $\mathbb{P}(|A_k| \geq \varepsilon) \leq \varepsilon^{-2}\, \mathbb{E}\, A_k^2$, so $\sum_k \mathbb{P}(|A_k| \geq \varepsilon) < \infty$. Therefore, by the Borel-Cantelli lemma, we have $\mathbb{P}(\limsup\{|A_k| \geq \varepsilon\}) = 0$ and

$$\mathbb{P}(\exists N \in \mathbb{N},\ \forall n \geq N,\ |A_n| \leq \varepsilon) = 1.$$

A countable intersection of almost sure events is almost sure, it follows that,

$$\mathbb{P}(\forall \varepsilon \in \mathbb{Q}_+^*,\ \exists N \in \mathbb{N},\ \forall n \geq N\ |A_n| \leq \varepsilon) = 1.$$

This exactly means $A_k \to 0$ almost surely.

(d) What can you say about the random variable $QV_t(B)$? We have,

$$\lim_{\epsilon \to 0} \sup_{|\underline{t}| \leq \epsilon} \sum_{i=1}^{\#\underline{t}} (B_{t_i} - B_{t_{i-1}})^2 \geq \lim_{k \to \infty} \sum_{i=1}^{\#\underline{t}^{(k)}} (B_{t_i^{(k)}} - B_{t_{i-1}^{(k)}})^2 = t.$$

Therefore, the quadratic variation of the trajectories of the Brownian motion is almost surely $\geq t$.

(2) Show that almost surely the trajectories of the Brownian do not have bounded total variation, that is $\mathbb{P}(TV_t(B) = \infty) = 1$. (*Hint*: what can you say about the quadratic variation of a continuous function with finite total variation ?).

Let $f : [0, t] \to \mathbb{R}$ be a continuous function, assume that $TV_t(f) < +\infty$ and let us show that $QV_t(f) = 0$. Let $\alpha > 0$, there exists $\varepsilon > 0$ such that for every $t_1, t_2 \leq t$, if $|t_1 - t_2| \leq \varepsilon$ then $|f(t_1) - f(t_2)| \leq \alpha$. Let $\underline{t}$ be a partition with mesh size $\leq \varepsilon$ and length n, we have

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2 \leq \sup_{1 \leq j \leq} |f(t_j) - f(t_{j-1})| \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \leq \alpha| \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

Taking the sup over all such partitions and letting the meshsize go to 0, we obtain $QV_t(f) \leq \alpha TV_t(f)$. This is true for all $\alpha > 0$, so $QV_t(f) = 0$. Since the trajectories of B are continuous and $QV_t(B) \geq t > 0$, we must have $TV_t(B) = +\infty$.

Exercise 7 — *A weaker condition for the first Wald's lemma.*
We wish to show that when T is a stopping time with $\mathbb{E}[T^{1/2}] < \infty$, Wald's lemma still applies and $\mathbb{E}[B_T] = 0$

(1) Define $\tau := \min\{k : 4^k \geq T\}$. Set $M(t) := \max_{[0,t]} B$ and $X_k := M(4^k) - 2^{k+2}$. Show that (X_k) is a supermartingale for the filtration $(\mathcal{F}_{4^k})_k$, and that τ is a stopping time. Define $\tau := \min\{k : 4^k \geq T\}$. Set $M(t) := \max_{[0,t]} B$ and

$$\mathbb{E}[X_{k+1} - X_k \mid \mathcal{F}_{4^k}] = \mathbb{E}[M(4^{k+1}) - M(4^k) \mid \mathcal{F}_{4^k}] - 4 \times 2^k.$$

Since we know that almost surely $M(4^{k+1}) - M(4^k) \leq |B_{4^{k+1}} - B_{4^k}|$ which is independent of $\mathcal{F}_{4^k}$ and distributed like $|B_{4^{k+1}-4^k}|$, then

$$\mathbb{E}[X_{k+1} - X_k \mid \mathcal{F}_{4^k}] \leq \mathbb{E}[|B_{4^{k+1}-4^k}|] - 4 \times 2^k = \sqrt{3 \times 4^k}\, \mathbb{E}[|B_1|] - 4 \times 2^k.$$

A simple application of Cauchy-Schwarz or Jensen gives $\mathbb{E}[|B_1|] \leq \sqrt{\mathbb{E}[|B_1|^2]} = 1$, and the expectation above is bounded by 0. If we consider τ, we have the equality of events $\{\tau \leq k\} = \{4^k \geq T\}$, which belongs to $\mathcal{F}_{4^k}$. So τ is a $(\mathcal{F}_{4^k})_k$-stopping time.

(2) Show that $\mathbb{E}[M(4^\tau)] < \infty$ and conclude. Let $n \geq 0$. $\mathbb{E}[M(4^\tau \wedge 4^n)] = \mathbb{E}[X_{\tau\wedge n}] + \mathbb{E}[2^{\tau\wedge n+2}] \leq \mathbb{E}[X_0] + 8\,\mathbb{E}[T^{1/2}]$, where we have used the supermartingale property at the bounded stopping time $\tau \wedge n$ and the fact that $4^\tau \leq 4T$. By monotone convergence $M(4^\tau)$ is integrable so $\max_{[0,T]} B \leq M(4^\tau)$ too. By reversal, $-\min_{[0,T]} B$ is integrable also, and this provides an integrable random variable that bounds $B_{t\wedge T}$ for every t. So the optional stopping theorem applies and $\mathbb{E}[B_T] = 0$.

(3) Show that when T is the hitting time of 1, then $\mathbb{E}[T^\alpha] < \infty$ for all $\alpha < 1/2$, yielding that our result is in some sense optimal. The law of T is the law of $1/|B_1|^2$. If $\alpha < 1/2$, then $t^\alpha \times t^{-3/2} e^{-1/(2t)}$ is $o(e^{-1/(2t)})$ (so it's integrable) near 0, and is $O(t^{-1-(1/2-\alpha)})$ near infinity, so is integrable too. In this case, $E[T^\alpha] < \infty$. For $\alpha = 1/2$, the function is no longer integrable at $+\infty$.