Master 1 — Spring 2025
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TD11 : Donsker's Invariance Principle and Arcsine Laws

Exercise 1 — *Another arcsine law.*
Let B be a Brownian motion on $[0, 1]$.

(1) Let $[a_1, b_1]$ and $[a_2, b_2]$ be two non overlapping intervals ($b_1 \leq a_2$). Show that almost surely the maximum value of B on $[a_1, b_1]$ and $[a_2, b_2]$ are different.

We start by arguing that we can choose $b_1 < a_2$. By Markov's property, $(B_{t+a_2} - B_{a_2})_{t\geq 0}$ is a Brownian motion so, almost surely, it takes positive values close to a_2. in other word, almost surely a_2 is not a maximum of B on $[a_2, b_1]$ and there exists n such that the maximum on $[a_2, b_2]$ coincides with the maximum on $[a_2 + 1/n, b_2]$. We let m_i denote the maximum on $[a_i, b_i]$. The event $m_1 = m_2$ is the same as the event,

$$B_{a_2} - B_{b_1} = m_1 - B_{b_1} - (m_2 - B_{a_2}).$$

In addition, by the Markov property $m_1 - B_{b_1}$ is independent of $B_{a_2} - B_{b_1}$ and for the same reason $m_2 - B_{a_2}$ is also independent of $B_{a_2} - B_{b_1}$. Conditioning, with respect to $m_1 - B_{b_1}$ and $m_2 - B_{a_2}$ in the previous display we obtain that the right-hand side is constant and the left-hand side has a continuous law, a contradiction.

(a) Show that the global maximum of B on $[0, 1]$ is attained at a unique point $M \in [0, 1]$.
According to the previous question, the following holds,

$$\mathbb{P}(\forall q \in \mathbb{Q} \cap [0, 1], \text{ the maximum of B on } [0, q] \text{ and } [q, 1] \text{ are distinct}) = 0.$$

By contradiction, assume that B attains is attained at two distinct points $t_1 < t_2$, then there exists $q \in \mathbb{Q} \cap [0, 1]$ such that $t_1 < q < t_2$ and the maximum values of B on $[0, q]$ and $[1, q]$ are equal. Therefore, the even "the global maximum of B is attained at at least two distinct points" is included in the complement of an almost sure event, so the desired result is proven.

(b) Every local maximum of B is a strict local maximum. Almost surely, any pair of disjoint interval with rational endpoints have different maximum. If B has a non-strict maximum, we can build two such intervals with same maximum.

(c) The set of points where the local maxima are attained is dense and countable. Since every local maximum is strict, the set of maximizer is discrete, hence countable. In addition, almost surely the maximum over any non-degenerate interval with rational endpoints is not attained at an endpoint, so every such interval contains a local maximum and the set of maximizers is dense.

(2) Show that for every $s \in [0, 1]$, $\mathbb{P}(M \leq s) = \frac{2}{\pi} \arcsin(\sqrt{s})$.

Let $s \in [0,1]$, we have

$$\begin{aligned}\mathbb{P}(M \leq s) &= \mathbb{P}(\max_{[0,s]} B_u > \max_{[s,1]} B_u)\\ &= \mathbb{P}(\max_{[0,s]} B_u - B_s > \max_{[s,1]} B_u - B_s)\\ &= \mathbb{P}(M_1(s) > M_2(s)),\end{aligned}$$

where $M_1(s)$ is the maximum on $[0,s]$ of the Brownian motion $B^1_u = B_{s-u} - B_s$ and $M_2(1-s)$ is the maximum on $[s,1]$ of the Brownian motion $B^2_u = B_{s+u} - B_s$. By the reflection principle, $M_1(s)$ is distributed like $|B_1(s)|$ and $M_2(1-s)$ like $|B_2(1-s)|$. So,

$$\mathbb{P}(M \leq s) = \mathbb{P}(|B_1(s)| \geq |B_2(s)) = \mathbb{P}(\frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} \leq \sqrt{s}),$$

where Z_1 and Z_2 are independent standard normal random variables. If we write the point (Z_1, Z_2) in polar coordinates, its angle with the origin θ is uniformly distrusted in $[0, 2\pi]$ and $\frac{|Z_2|}{\sqrt{Z_1^2+Z_2^2}} = |\sin(\theta)|$ which finishes the proof.

Exercise 2 — *Yet another arcsine law.*
Let $(X_k)_{k\geq 1}$ be a sequence of iid standard random variables, let $(S_n)_{n\geq 0}$ be the random walk associated to $(X_k)_{k\geq 1}$. Let

$$N_n = \max\{k \in \{1, \dots, n\}, S_k S_{k-1} \leq 0\}$$

be the last sign change of (S_k) before time n. Given $f \in \mathcal{C}([0,1])$, let

$$G(f) = \sup\{t \in [0,1], f(t) = 0\}$$

denote its last zero. Let U denote the set of functions $f \in \mathcal{C}([0,1])$ such that $f(1) \neq 0$ and for every $t \in [0,1]$, if $f(t) = 0$ then for every $\varepsilon > 0$, the function f takes positive and negative values in $[t-\varepsilon, t+\varepsilon]$.

(1) Recall how to define $S^*_n \in \mathcal{C}([0,1])$ using the trajectory $(S_k)_{1\leq k\leq n}$ of the random walk. Given a Brownian motion B, what is the law of $G(B)$?

According to the arcsine law, $G(B)$ is arcsine distributed. Given a trajectory of S on $\{1, \dots, n\}$ we can define a continuous function on $[0,1]$ by considering the piecewise linear function whose value at $\frac{k}{n}$ is $\frac{S_k}{\sqrt{n}}$. Those functions are the function appearing in Donsker's invariance principle.

(2) Show that for every $f \in U$, the function G is continuous at f.

Let $f \in U$, and let $(f_n)_n$ be a sequence of functions that converge to f uniformly on $[0,1]$. Let $t_n = G(f_n) \in [0,1]$ and let t be a limit point of some subsequence $(t_{\varphi(n)})_n$, let us show that $t = G(f)$. We have $f_n(t_n) = 0$, so letting $n \to \infty$ along a subsequence, we obtain $f(t) = 0$ and $t \leq G(f)$. By contradiction, assume $t < G(f)$ then there exists $s \in (t,1)$ such that $f(s) = 0$. Since $f \in U$, for every $\varepsilon > 0$, there exists $s^+_\varepsilon, s^-_\varepsilon \in [s-\varepsilon, s+\varepsilon]$ such that $f(s^+_\varepsilon) > 0$ and $f(s^+_\varepsilon) < 0$. Let us fix $\varepsilon > 0$ os that $s - \varepsilon > t$, for n large enough $f_{\varphi(n)}(s^+_\varepsilon) > 0$ and $f_{\varphi(n)}(s^+_\varepsilon) > 0$, so $f_{\varphi(n)}$ vanishes in $[s-\varepsilon, s+\varepsilon]$. Thus, $t_{\varphi(n)} = G(f_{\varphi(n)}) \geq s - \varepsilon > t$, letting $n \to \infty$ we discover that $t \geq s - \varepsilon < t$, a contradiction !

(3) Show that almost surely G is continuous at B. (*Hint: Exercise 4 of TD9 and Exercise 1 of TD11*)

Let us show tht almost surely $B \in U$. It is clear that almost surely $B_1 \neq 0$. Let us show that almost surely B changes sign in every neighborhood of its zeros. Assume this is not the case, then there exists t a zero of B, such that B does not change sign in a neighborhood of t, then t is a local extremum of B, by Question 2.(b) of exercise 2 in TD11 it is a strict local extremum. This means that t would be an isolated zero of B. But according to Exercise 4 of TD9, almost surely the set of zeros of B has no isolated points. Therefore, $B \in U$ almost surely and G is continuous at B almost surely.

(4) Show that N_n/n converges in law to N, where $\mathbb{P}(N \leq x) = \frac{2}{\pi}\arcsin(x)$.

To apply Donsker's invariance principle we only need continuity of the functional on U, so we can apply it to G here. It is clear that $|N_n/n - G(S_n^*)| \leq 1/n$, so for any bounded continuous function $h : \mathbb{R} \to \mathbb{R}$, by dominated convergence we have.

$$\lim_{n\to\infty} \mathbb{E}\left(h\left(\frac{N_n}{n}\right) - h(G(S_n^*))\right) = 0.$$

In addition, by Donsker's invariance principle, $G(S_n^*) \to G(B)$ so by dominated convergence $\mathbb{E}\, h(G(S_n^*)) \to \mathbb{E}\, h(G(B))$. In conclusion, $\frac{N_n}{n}$ converges in law to $G(B)$ which is arcsine distributed.

Exercise 3 — *Maximum value of a random walk.*
Let $(X_k)_{k\geq 1}$ be a sequence of iid standard random variables, let $(S_n)_{n\geq 0}$ be the random walk associated to $(X_k)_{k\geq 1}$. Define,

$$M_N = \sup\{S_n, 0 \leq n \leq N\}.$$

Compute the limit in law of $M_N/\sqrt{N}$ as $N \to \infty$.
Let $g \in \mathcal{C}_b(\mathbb{R})$, define $G : \mathcal{C}([0,1]) \to \mathbb{R}$ by $G(f) = g(\max f)$. The function G is bounded and continuous with respect to the topology of uniform convergence. Let $S_n^* \in \mathcal{C}([0,1])$ be a rescaled and linearly interpolated version of S_n, we have $\max S_N^* = \max_{0\leq n\leq N} \frac{S_n}{\sqrt{N}} = \frac{M_N}{\sqrt{N}}$. Therefore, $\mathbb{E}\, g(\frac{M_N}{\sqrt{N}}) = \mathbb{E}\, G(S_N^*)$ and by Donsker's invariance principle,

$$\mathbb{E}\, g\left(\frac{M_N}{\sqrt{N}}\right) \to \mathbb{E}\, G(B) = \mathbb{E}\, g(\max_{0\leq t\leq 1} B_t).$$

Finally, according to the reflection principle $\max_{0\leq t\leq 1} B_t \overset{(d)}{=} |B_1|$. So $M_N/\sqrt{N}$ converges in law to the absolute value of a standard normal.