Master 1 — Spring 2025
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TD10 : Continuous Time Martingales

Exercise 1 — *Hitting time of a line.*
Let $a \geq 0$ and $b \in \mathbb{R}$, define $T = \inf\{t \geq 0, B_t = at + b\}$. Compute $\mathbb{P}(T < \infty)$.
We can restrict to $b > 0$ using the intermediate value theorem. Let $X_t = e^{2aB_t - 2a^2 t}$ be the exponential martingale for $\lambda = 2a$. For every $t \leq T$, $X_t \leq e^{2ab}$, so the stopped martingale X^T is bounded by a constant. By the optionnal stopping theorem,

$$1 = \mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_T \mathbf{1}_{\{T \leq n\}}] + \mathbb{E}[X_n \mathbf{1}_{\{T > n\}}]$$

By monotone convergence $\lim_{n\to\infty} \mathbb{E}[X_T \mathbf{1}_{\{T\leq n\}}] = \mathbb{E}[X_T \mathbf{1}_{\{T<\infty\}}]$. To treat the second term, recall that $\lim_{t\to\infty} \frac{B_t}{t} = 0$ almost surely, So $0 \leq X_n \mathbf{1}_{\{T>n\}} \leq e^{2at(\frac{B_t}{t} - a)} \to 0$ almost surely. In addition $0 \leq X_n \mathbf{1}_{\{T>n\}} \leq e^{2ab}$, so by dominated convergence, $\lim_{n\to\infty} \mathbb{E}[X_n \mathbf{1}_{\{T>n\}}] = 0$. So letting $n \to \infty$, we obtain $1 = \mathbb{E}[X_T \mathbf{1}_{T<\infty}]$, but $X_T = e^{2ab}$. Thus, $\mathbb{P}(T < \infty) = e^{-2ab}$.

Exercise 2 — *All hypotheses matter.*
Let B be a Brownian motion and S, T two stopping times such that $S \leq T < \infty$ almost surely.

(1) Show that if $\mathbb{E}[S] < \infty$ and $\mathbb{E}[T] < \infty$, then $\mathbb{E}[B_S^2] \leq \mathbb{E}[B_T^2]$.
According to Wald's second lemma, if T is a stopping time with finite first moment, we have $\mathbb{E}[B_T^2] = \mathbb{E}[T]$. Here it is clear that $\mathbb{E}[S] \leq \mathbb{E}[T]$, so the result follows.

(2) Find two stopping times S and T with $\mathbb{E}[S] < \infty$, such that $\mathbb{E}[B_S^2] > \mathbb{E}[B_T^2]$. Let $S = 3$, and let

$$T = \inf\{t \geq 3, B_t = 0\},$$

Be the first hitting time of 0 after time 3. Clearly S and T are stopping times and $\mathbb{E}[S] = 3 < \infty$ and by definition $S \leq T$, $\mathbb{E}[B_3^2] = 1$ and $\mathbb{E}[B_T^2] = 0$. Let us show that $\mathbb{P}(T < \infty) = 1$, we have

$$\begin{aligned}\{T = \infty\} &= \{\forall t \geq 3, B_t \neq 0\} \\ &= \{\forall t \geq 0, B_{t+3} \neq 0\} \\ &= \{\forall t \geq 0, B_t^{(3)} \neq -B_3\}.\end{aligned}$$

Given a process X and $a \in \mathbb{R}$ we denote $T_a(X)$ the first hitting of a by X. For every $a \in \mathbb{R}$, we have $T_a(B) < \infty$ almost surely. It follows,

$$\begin{aligned}\mathbb{P}(T < \infty) &= \mathbb{E}[\mathbf{1}_{\{T<\infty\}}] \\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{T<\infty\}}|B_3]] \\ &= \mathbb{E}[\mathbb{P}(T_{-B_3}(B^{(3)}) < \infty|B_3)] \\ &= \mathbb{E}[1] \\ &= 1.\end{aligned}$$

Exercise 3 — *Brownian gambler's ruin.*
For any $c \in \mathbb{R}$, we let

$$T_c := \inf\{t \geq 0 \ : \ B_t = c\}$$

be the hitting time of c by $(B_t)_{t\geq 0}$. Let $a, b > 0$, we let $T := T_{-a} \wedge T_b$ be the hitting time of $\{-a, b\}$ by $(B_t)_{t\geq 0}$.

(1) What is the law of B_T? The random variable B_T is supported on $-a, b$. Let $p = \mathbb{P}(B_T = -a) = \mathbb{P}(T = T_{-a})$. The stopped process B^T is bounded by the maximum value between a and b. Applying the optional stopping time theorem, we obtain,

$$0 = \mathbb{E}[B_0] = \mathbb{E}[B_T] = p(-a) + (1-p)b.$$

Solving for p, we obtain $p = \frac{b}{b+a}$.

(2) Compute $\mathbb{E}[T]$. Consider the quadratic martingale $X_t = B_t^2 - t$, by the martingale property applied to M_t^T, we have

$$\mathbb{E}[(B_t^T)^2] = \mathbb{E}[T \wedge t].$$

The process B^T is bounded by the maximum of a and b. Since T is almost surely finite, by dominated convergence we have $\lim_{t\to\infty} \mathbb{E}(B_t^T)^2 = \mathbb{E}\, B_T^2 = ab$ By monotone convergence, $\mathbb{E}[T \wedge t] \to \mathbb{E}[T]$ so $\mathbb{E}[T] = ab$.

Exercise 4 — *Exponential martingale and computations.*
Let B be a Brownian motion, we recall that for every $\lambda \in \mathbb{R}$, the process $(e^{\lambda B_t - t\lambda^2/2})_{t\geq 0}$ is a martingale, called the exponential martingale. We let for any $a > 0$,

$$T_{a+} := \inf\{t \geq 0 \ : \ B_t > a\}$$

(1) For every $a > 0$ and $\mu \geq 0$, compute the Laplace transform $\mathbb{E}[e^{-\mu T_{a+}}]$. (*Hint:* use the exponential martingale). To lighten notations, we write T in place of T_{a+} in this question. Let $X_t = e^{\lambda B_t - t\lambda^2/2}$, the stopped martingale X^T is bounded by $e^{\lambda a}$. The optional stopping time theorem implies $1 = \mathbb{E}[X_T] = e^{\lambda a}\,\mathbb{E}[e^{-\frac{\lambda^2}{2}T}]$. So for every $\mu \geq 0$, $\mathbb{E}\, e^{-\mu T} = e^{-a\sqrt{2\mu}}$.

(2) Let $(B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion. For every $a \geq 0$, we let

$$C_a := B^{(2)}_{T^{(1)}_{a+}}.$$

(a) Show that for any $b > 0$, the process $C^{(b)} = (C_{b+a} - C_b)_{a\geq 0}$ is independent of $\mathcal{F}_{T_{b+}}$ and has the same law as $(C_a)_{a\geq 0}$. Deduce that $(C_a)_{a\geq 0}$ is a Markov process and give its transition kernel. We have, $C^{(b)}(B) = C(B^{(T^{(1)}_{b+})})$. So, by the Markov property for B, the result follows. The process C is a Markov process with kernel,

$$P_t(x, A) = \mathbb{P}(C_t \in A).$$

(b) Show that $(e^{\lambda(B_t^{(1)} + iB_t^{(2)})})_{t\geq 0}$ is a complex martingale, and deduce the characteristic function of C_a for $a > 0$ fixed. Let M denote the process in question, Since B has independent and stationary increments, we have

$$\begin{aligned}
\mathbb{E}[M_{t+s}|\mathcal{F}_t] &= \mathbb{E}[\exp(\lambda((B^{(1)}_{t+s} - B^{(1)}_t) + i(B^{(2)}_{t+s} - B^{(2)}_t)))\exp(\lambda(B^{(1)}_t + iB^{(2)}_t))|\mathcal{F}_t] \\
&= \mathbb{E}[\exp(\lambda((B^{(1)}_{t+s} - B^{(1)}_t) + i(B^{(2)}_{t+s} - B^{(2)}_t)))|\mathcal{F}_t]\exp(\lambda(B^{(1)}_t + iB^{(2)}_t)) \\
&= 1 \times M_t.
\end{aligned}$$

Applying the optional stopping theorem to M and $T^{(1)}_{a+}$ (the stopped martingale is bounded by $e^{\lambda a}$, we obtain

$$1 = \mathbb{E}[M_{T^{(1)}_{a+}}] = \mathbb{E}[\exp(\lambda(a + iC_a))]$$

Therefore,

$$\mathbb{E}[e^{i\lambda C_a}] = e^{-\lambda a}.$$

Finally, note that $-C_a \overset{(d)}{=} C_a$ so applying the above formula (only valid for $\lambda \geq 0$) we obtain,

$$\mathbb{E}[e^{i\lambda C_a}] = e^{-|\lambda| a}.$$

(c) Compute the distribution of C_a. We recognize the Laplace transform of the Cauchy Law with parameter a, whose density is,

$$f(x) = \frac{1}{\pi a}\frac{1}{1 + \left(\frac{x}{a}\right)^2}.$$

Exercise 5 — *Exponential Martingale.*
Show that if $(X_t)_{t\geq 0}$ is a process such that for any $\lambda \in \mathbb{R}$, $(e^{\lambda X_t - t\lambda^2/2})_{t\geq 0}$ is a continuous martingale started from 1, then $(X_t)_{t\geq 0}$ has the law of a Brownian motion.

Assume that for every $\lambda \in \mathbb{R}$ $M_t(\lambda) = e^{\lambda X_t - t\lambda^2/2}$ defines a continuous martingale started from 1. We are going to show that X is Brownian motion using Lévy's characterization. We have,

$$X_t = \log M_t(1) + t/2,$$

so X has continuous trajectories and $X_0 = 0$. In addition, for every, $s < t$ we have

$$\begin{aligned}
\mathbb{E}[e^{\lambda(X_t - X_s)}] &= \mathbb{E}\left[\mathbb{E}\left[e^{\lambda(X_t-X_s)}|\mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\lambda X_t - t\lambda^2/2}e^{-(\lambda X_s - s\lambda^2/2)}|\mathcal{F}_f\right]\right]e^{(t-s)\lambda^2/2} \\
&= \mathbb{E}\left[\mathbb{E}\left[M_t(\lambda)|\mathcal{F}_s\right]M_s(\lambda)^{-1}\right]e^{(t-s)\lambda^2/2} \\
&= \mathbb{E}[M_s M_s^{-1}]e^{(t-s)\lambda^2/2} \\
&= e^{(t-s)\lambda^2/2}.
\end{aligned}$$

This means that $X_t - X_s$ is a centered Gaussian random variable with variance $t - s$. Furthermore, a similar computation yields for $u < s < t$

$$\begin{aligned}
\mathbb{E}[M_t(\lambda)M_s^{-1}(\lambda)M_u(\mu)] &= \mathbb{E}[\mathbb{E}[M_t(\lambda)M_s^{-1}(\lambda)|\mathcal{F}_u]M_u(\mu)] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{E}[M_t(\lambda)M_s^{-1}(\lambda)|\mathcal{F}_s]|\mathcal{F}_u]M_u(\mu)] \\
&= \mathbb{E}[\mathbb{E}[\mathbb{E}[M_t(\lambda)|\mathcal{F}_s]M_s^{-1}(\lambda)|\mathcal{F}_u]M_u(\mu)] \\
&= \mathbb{E}[\mathbb{E}[M_s(\lambda)M_s^{-1}(\lambda)|\mathcal{F}_u]M_u(\mu)] \\
&= \mathbb{E}[M_u(\mu)].
\end{aligned}$$

This means, that

$$\mathbb{E}[e^{\lambda(X_t-X_s)+\mu X_u}] = \mathbb{E}[e^{\lambda(X_t-X_s)}]\,\mathbb{E}[e^{\mu X_u}].$$

In conclusion X starts from 0, has continuous trajectories, has centered normal independent increments with variance $t - s$, thus X is a standard Brownian motion.

Exercise 6 — *Martingales derived from B.*
Let B be a Brownian motion. For $n \geq 0$, we define the n-th Hermite polynomial H_n by,

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}.$$

We equip the vector space $\mathbb{R}[X]$ of real polynomials with the scalar product,

$$P \cdot Q = \int_{\mathbb{R}} P(x)Q(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx.$$

(1) Show that $(H_n)_{n\geq 0}$ is an orthogonal family in $\mathbb{R}[X]$.
    This is a classical result which follows from a few integrations by part.
(2) Show that for every $\lambda, b \in \mathbb{R}$, $e^{\lambda b - \frac{\lambda^2}{2}} = \sum_{n\geq 0}\frac{H_n(b)}{n!}\lambda^n$.

Define $f(\lambda, x) = e^{-\frac{1}{2}(\lambda - x)^2}$, we have

$$f(\lambda, x) = \sum_{n \geq 0} \frac{\partial_\lambda^n f(0, x)}{n!} \lambda^n.$$

In addition,

$$\partial_\lambda^n f(0, x) = \frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} e^{-\frac{1}{2}(\lambda - x)^2} = (-1)^n \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2} = e^{-\frac{x^2}{2}} H_n(x).$$

Therefore,

$$\sum_{n \geq 0} \frac{H_n(x)}{n!} \lambda^n = e^{\frac{x^2}{2}} f(\lambda, x) = e^{\frac{x^2}{2}} e^{-\frac{1}{2}(\lambda - x)^2} = e^{\lambda x - \frac{\lambda^2}{2}}.$$

(3) Show that for every $n \geq 0$, the process $(t^{n/2} H_n\left(\frac{B_t}{\sqrt{t}}\right))_{t \geq 0}$ is a martingale.

This can be seen as a consequence of the fact that the exponential martingale is a martingale and this property is preserved when differentiating with respect to λ.