Master 1 — Spring 2025
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TD9 : Stopping Times

Exercise 1 — *Warm-up.*
Let B be a Brownian motion, using the fact that B is a Gaussian process, show that B has the simple Markov property. That is for every $s \geq 0$ show that $(B_{t+s} - B_s)_{t\geq 0}$ is a Brownian motion independent from $(B_t)_{t\leq s}$.
It is easly checked that $(B_{t+s} - B_s)_{t\geq 0}$ is a Gaussian process started from 0 with continuous trajectories and with covariance of the form $\min\{t_1, t_2\}$. Hence, $(B_{t+s} - B_s)_{t\geq 0}$ is a Brownian motion. Furthermore, the covariance between $(B_{t+s} - B_s)_{t\geq 0}$ and $(B_t)_{0\leq\leq s}$ is 0, so those two Gaussian processes are independent.

Exercise 2 — *Counter-example.*
Let B be a Brownian motion and let

$$T = \inf\{t \geq 0, B_t = \max_{s\in[0,1]} B_s\},$$

be the first hitting time by B of the maximum of B on $[0,1]$, is T a stopping time ?
At an intuitive level we see that to have $T = t$, we need to know B will not exceed the value B_t on $[1, t]$. Hence, we can guess that T is not a stopping time; let us prove this rigorously. By contradiction assume that T is a stopping time, then $\tilde{B}_t = B_{T+t} - B_T$ is a Brownian motion. By definition of T, $\tilde{B}$ is non-positive on $[0, 1-T]$. If $T < 1$ we reach a contradiction, as we've just built a non-trivial interval on which B has constant sign. Otherwise, we have $T = 1$ a.s., this is also not possible as we would have for every $t \in [0,1]$, $B_t \leq B_1$ and the time reversed Brownian motion $(B_1 - B_{1-t})_{t\in[0,1]}$ would have constant sign.

Exercise 3 — *Hölder regularity of Brownian trajectories.*
Let B be a Brownian motion, recall from Exercise 4 of TD8 that the process $X = (tB_{1/t})_{t\geq 0}$ is also a Brownian motion.

(1) Show that $\lim_{t\to\infty} \frac{B_t}{t} = 0$ a.s. and that for every $\varepsilon > 0$, $\lim_{t\to\infty} \frac{B_t}{t^{1/2+\varepsilon}} = 0$ a.s..
Setting $s = \frac{1}{t}$, we have $\lim_{t\to\infty} \frac{B_t}{t} = \lim_{s\to 0} X_s = 0$. In addition, since X is $1/2 - \varepsilon/2$-Hölder in a neighborhood of 0, we have

$$\lim_{t\to\infty} \frac{B_t}{t^{1/2+\varepsilon}} = \lim_{s\to 0} s^{\varepsilon}/2 \frac{X_s}{s^{\frac{1-\varepsilon}{2}}} = 0.$$

(2) Let (ξ_n) be a sequence of independent and identically distributed centered random variables with variance 1,
  (a) For every $K \in \mathbb{R}$, show that $\mathbb{P}(\limsup_{n\to\infty}\{\sum_{k=1}^{n} \xi_k \geq K\sqrt{n}\}) > 0$.

$$\begin{aligned}
\mathbb{P}(\limsup_{n\to\infty}\{\sum_{k=1}^n \xi_k \geq K\sqrt{n}\}) &= \mathbb{P}(\cap_N \cup_{n\geq N} \{\sum_{k=1}^n \xi_k \geq K\sqrt{n}\}) \\
&= \lim_{N\to\infty} \mathbb{P}(\cup_{n\geq N}\{\sum_{k=1}^n \xi_k \geq K\sqrt{n}\}) \\
&\geq \lim_{N\to\infty} \mathbb{P}(\{\sum_{k=1}^N \xi_k \geq K\sqrt{N}\}) \\
&= \mathbb{P}(\mathcal{N}(0,1) \geq K) \\
&> 0.
\end{aligned}$$

Where we have used the central limit theorem to go from 2nd to last to last line.

(b) Show that almost surely

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^n \xi_k = +\infty.$$

By Kolmogorov's 0-1 law, the probability of the event in the previous question is actually $= 1$. Since $\limsup\{f(n) \geq K\} = \{\limsup f(n) \geq K\}$, we have for every $K \geq 1$, $\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^n \xi_k \geq K$. Thus the result.

(3) Deduce from the previous question that,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty,$$

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t\to 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

Since $-B$ is also a Brownian motion, the second inequality in each display follows from the first. For the first eqaulity of the first display, observe that we can apply the result of the previous question to $\xi_n = B_n - B_{n-1}$, and obtain $\limsup_{n\to\infty} \frac{B_n}{\sqrt{n}} = +\infty$. But $\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} \geq \limsup_{n\to\infty} \frac{B_n}{\sqrt{n}}$, so the result follows. To get the first inequality of the second display, we deduce from the time inversion property by applying the first inequality of the first display to X.

(4) Recall that given a function $f : \mathbb{R}_+ \to \mathbb{R}$ and $t \geq 0$, we say that f is locally α-Hölder near t if there exists a neighborhood V of t in $\mathbb{R}_+$ such that for every $s \in V$, we have

$$|f(s) - f(t)| \leq |t - s|^\alpha.$$

(a) Show that almost surely B is not locally $1/2$-Hölder near 0.
This the content of the previous proposition.

(b) For every $t \geq 0$, define $L_t = \limsup_{s\to 0} \left|\frac{B_{t+s}-B_t}{\sqrt{s}}\right|$ and $A = \{t \in \mathbb{R}_+, L_t < +\infty\}$. Show that almost surely the set A is negligible with respect to the Lebesgue measure.
According to the previous question for every t, $L_t = \infty$ almost surely, so we have

$$\begin{aligned}\mathbb{E}\,\mathcal{L}^1(A) &= \mathbb{E}\int_0^\infty 1_{L_t<\infty}dt \\ &= \int_0^\infty \mathbb{P}(L_t < \infty)dt \\ &= 0.\end{aligned}$$

(c) Show that the set of points near which B is locally 1/2-Hölder is almost surely negligible with respect to the Lesbesgue measure. This set is contained in the set the set of the privous question.

Exercise 4 — *The set of zeros of B is perfect.*
Recall that for every $S \subset \mathbb{R}$ we say that $x \in S$ is an isolated point if there exists a neighborhood V of x in $\mathbb{R}$ such that $V \cap S = \{x\}$. Let B be a Brownian motion, and

$$Z = \{t \geq 0 : \ B_t = 0\}.$$

(1) Show that the following events are almost sure,
  (a) Z is infinite and is a closed set. Z is the zero set of a continuous function, so it is closed. We know that the trajectories of the Brownian change sign infinitely many times close to 0 so Z is infinite.
  (b) Z has Lebesgue measure 0. Let $\mathcal{L}^1$ denote the Lesbesgue measure on $\mathbb{R}_+$. let $T > 0$, we have

$$\begin{aligned}\mathbb{E}\,\mathcal{L}^1(Z\cap[0,T]) &= \mathbb{E}\int_0^T \mathbf{1}_{t\in Z}dt \\ &= \int_0^T \mathbb{E}\,\mathbf{1}_{t\in Z}dt \\ &= \int_0^T \mathbb{P}(t\in Z)dt \\ &= \int_0^T \mathbb{P}(B_t = 0)dt \\ &= 0.\end{aligned}$$

  For every $T > 0$, almost surely $\mathcal{L}^1(Z\cap[0,T]) = 0$, taking union over $T \in \mathbb{N}$, yields almost surely $\mathcal{L}^1(Z) = 0$.
  (c) Z has no isolated points. Let $t \in Z$, define $R_t = \inf\{s > 0, B_{s+t} - B_t = 0\}$ the hitting time of 0 by $B^{(t)}$. Since a Brownian motion vanishes infinitely many

times close to 0, according to the simple Markov property, we have $R_t = 0$ almost surely. So almost surely, for every $\varepsilon > 0$, there exists $s \in (0, \varepsilon)$ such that $B_{t+s} - B_t = 0$, that is $t + s \in Z$. so t is not isolated in Z.

(2) ($\star$) Make the definition of random closed set rigorous by defining a σ-algebra on the set of closed subsets of $\mathbb{R}_+$. Let $\mathbb{F}$ denote the set of closed subsets of $\mathbb{R}_+$, the Hausdorff distance

$$d(F, G) = \max\left\{\sup_{f\in F} d(f, G), \sup_{g\in G} d(g, F)\right\},$$

defines a metric on $\mathbb{F}$ and one can define a σ-algebra on $\mathbb{F}$ by choosing the Borel σ-algebra associated to d (the σ-algebra generated by open sets of $(\mathbb{F}, d)$).

Exercise 5 — *Another counter-example.*
Let $X_t = AB_t$ where B is a Brownian motion started from 1 and A an independent uniform random variable in $\{-1, 1\}$.

(1) Show that X is a Markov process and give its transition kernel.
(2) Show that it does not verify the strong Markov property.

Exercise 6 — *Brownian motion on the circle.*
Define a Brownian motion on the circle $\mathbb{S}^1$ by setting $X_t = e^{iB_t}$ for $t \geq 0$. What is the distribution of the last point hit by X in $\mathbb{S}^1$?