ENS de Lyon — Mathematic department | Master 1 — Spring 2025
Stochastic processes | V. Issa & E. Jacob

TD6 : Invariant Measures

Exercice 1 — *Birth and death processes II.*
We consider the pure jump Markov process with values in $\mathbb{N}$ and intensity matrix Q given by:

$$q_{i,j} = \begin{cases} \beta_i & \text{si } j = i+1 \\ \delta_i & \text{si } j = i-1 \\ -\beta_i - \delta_i & \text{si } j = i \neq 0 \\ -\beta_i & \text{si } j = i = 0 \\ 0 & \text{sinon,} \end{cases}$$

where the β_i et δ_i are assumed to be positive (so > 0). We let $r_n = \frac{1}{\beta_n} + \sum_{k=0}^{n-1} \frac{\delta_{k+1} \dots \delta_n}{\beta_k \dots \beta_n}$, and we consider x nonnegative bounded solution to the equation $Qx = x$.

(1) Show $x = 0$ iff $x_0 = 0$, and in the case of a nonzero solution is nonzero, show the sequence (x_n) is increasing.
The assumption $Qx = x$ means that coordinates of x satisfy,

$$\begin{cases} -\beta_0 x_0 + \beta_0 x_1 = x_0 \\ \delta_i x_{i-1} - (\beta_i + \delta_i) x_i + \beta_i x_{i+1} = x_i \text{ for } i \geq 1. \end{cases}$$

This can be recasted into,

$$\begin{cases} x_1 = \left(1 + \frac{1}{\beta_0}\right) x_0 \\ x_{i+1} = \left(1 + \frac{1}{\beta_i}\right) x_i + \frac{\delta_i}{\beta_i}(x_i - x_{i-1}) \text{ for } i \geq 1. \end{cases}$$

If $x_0 = 0$, then $x_1 = 0$ by the first equation. It follows by induction on i using the second equation thay $x_i = 0$ for every $i \geq 0$. If $x_0 \neq 0$, then $x_1 > x_0 > 0$ and we can show by induction on i that $0 < x_i < x_{i+1}$.

(2) For $i \in \mathbb{N}$, Show that we have $x_i + r_i x_0 \leq x_{i+1} \leq (1 + r_i) x_i$. We will reuse the recurrence equation of (1) and show the result by induction on i. We start by

showing a recurrence relation for $(r_i)_{i\geq 1}$. Let $i \geq 1$, we have

$$\begin{aligned}\frac{\delta_i}{\beta_i} r_{i-1} &= \frac{\delta_i}{\beta_i}\left(\frac{1}{\beta_{i-1}} + \sum_{k=0}^{i-2} \frac{\delta_{k+1}\dots\delta_{i-1}}{\beta_k \dots \beta_{i-1}}\right) \\ &= \frac{\delta_i}{\beta_i \beta_{i-1}} + \sum_{k=0}^{i-2} \frac{\delta_{k+1}\dots\delta_i}{\beta_k\dots\beta_i} \\ &= \sum_{k=0}^{i-1} \frac{\delta_{k+1}\dots\delta_i}{\beta_k\dots\beta_i} \\ &= r_i - \frac{1}{\beta_i}.\end{aligned}$$

We have shown,

$$r_i = \frac{1+\delta_i r_{i-1}}{\beta_i}.$$

We are now ready for the induction. At $i = 0$, we have $r_0 = 1/\beta_0$ so $x_0 + \beta_0 x_0 \leq x_1 \leq (1 + \frac{1}{\beta_0})x_0$ holds and is in fact an equality. Let $i \geq 1$ and assume that, $x_{i-1} + r_{i-1}x_0 \leq x_i \leq (1 + r_{i-1})x_i$, that is

$$r_{i-1}x_0 \leq x_i - x_{i-1} \leq r_{i-1}x_i.$$

In addition the recurrence relation $x_{i+1} = \left(1 + \frac{1}{\beta_i}\right) x_i + \frac{\delta_i}{\beta_i}(x_i - x_{i-1})$ can be rewritten as

$$x_{i+1} - x_i = \frac{1}{\beta_i} x_i + \frac{\delta_i}{\beta_i}(x_i - x_{i-1})$$

Combining the result from the last two displays, we obtain

$$\frac{1}{\beta_i} x_i + \frac{\delta_i}{\beta_i} r_{i-1}x_0 \leq x_{i+1} - x_i \leq \frac{1}{\beta_i} x_i + \frac{\delta_i}{\beta_i} r_{i-1}x_i$$

Using our recurrence relation on r_i, we see that the RHS of the above display is $= r_i x_i$, in addition since $x_i \geq x_0$, the LHS is lower bounded by $\frac{1+\delta_i r_{i-1}}{\beta_i} x_0 = r_i x_0$. We have proven,

$$x_i + r_i x_0 \leq x_{i+1} \leq (1 + r_i)x_i.$$

(3) We say that the process doesn't explode when for every $i \in \mathbb{N}$, the probability of explosion starting from i is 0. Show that the process doesn't explode if and only $\sum r_n = +\infty$. By (2), given x any nonnegative solution of $Qx = x$, we have or every $i \geq 0$,

$$x_0 \sum_{k=0}^{i-1} r_k \leq x_i \leq x_0 \prod_{k}^{i-1}(1 + r_k).$$

We will use the fact that $\prod_{1\leq k\leq i}(1 + r_k)$ converges to a finite limit as $i \to \infty$ if and only if $\sum r_k$ converges. By a Theorem from the course, the process explodes with probability 0 from any starting point if and only $Qx = x$ admits a unique nonnegative bounded solution. Assume that $\sum r_i$ diverges, let x be a nonzero, nonnegative solution of $Qx = x$, by (1) we must have $x_0 \neq 0$. It follows from the previous display that x is unbounded. therefore the only bounded nonnegative solution of $Qx = x$ is $x = 0$ and the process doesn't explode. Conversely, assume that $\sum r_i$ converges, let x be the vector defined by $x_0 = 1$,$x_1 = 1 + \frac{1}{\beta_0}$ and $x_{i+1} = (1 + \frac{1}{\beta_i})x_i + \frac{\delta_i}{\beta_i}(x_i - x_{i-1})$. By construction, x is a nonzero solution of $Qx = x$ and $x_0 \neq 0$, so by (1) $x_i > x_0 > 0$, so x is nonnegative. Finally, $\prod(1 + r_k)$ converges and the partial product upper bound x_i, so x is bounded. We have built $x \neq 0$ a nonnegative bounded solution of $Qx = x$, so by the criterion of the course the process explodes.

Let $\lambda_i > 0$ and $p \in (0, 1)$ set $q = 1 - p$ and consider X the continuous time Markov chain with intensity matrix Q with parameters $\beta_i = p\lambda_i$ and $\delta_i = q\lambda_i$.

(4) Show that the equation $\mu Q = 0$ has a unique solution up to multiplicative constant, assuming that $\mu_0 = \frac{1}{\lambda_0}$, give an explicit expression of μ_i for every $i \in \mathbb{N}$.

Asssume that $\mu Q = 0$, then μ satisfies,

$$\begin{cases} -\lambda_0 p\mu_0 + \lambda_1 q\mu_1 = 0 \\ \lambda_{i-1} p\mu_{i-1} - \lambda_i\mu_i + \lambda_{i+1} q\mu_{i+1} = 0 \text{ for } i \geq 1. \end{cases}$$

So the sequence $(\mu_i)_i$ satisfies a second order recurrence relation. It is fully determined by μ_0 and $\mu_1 = \frac{1}{\lambda_1}\frac{p}{q}\lambda_0\mu_0$. So μ is unique up to multiplicative constant. Assuming that $\mu_0 = \frac{1}{\lambda_0}$, we can check that

$$\mu_i = \frac{1}{\lambda_i}\left(\frac{p}{q}\right)^i.$$

(5) Assume that there exists $\lambda > 0$ such that for every $i \in \mathbb{N}$, $\lambda_i = \lambda$ and that $p < 1/2$. Show that X doesn't explode, that X admits a unique invariant probability measure and describe the set of invariant measures of X.

Under the above assumption we can check that X doesn't explode two ways. First the diagonal coefficients of Q are lower bounded. Or, using the criterion we just proved by checking that the sequence $(r_n)_n$ is not summable. Either way X doesn't blow up. with the assumption that $p < 1/2$, we have $\frac{p}{q} < 1$, so any solution of $\mu Q = 0$ has finite mass. Therefore invariant measures are exactly the solutions of $\mu Q = 0$. Among the solution of $\mu Q = 0$ the only probability measure is,

$$\mu_i = \left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^i.$$

Therefore, the measure defined above is the unique invariant probability measure of X. This a "geometric law" with parameters p/q (not quite be cause it models the number of failures until the first success).

(6) Assume that there exists $\lambda > 0$ such that for every $i \in \mathbb{N}$, $\lambda_i = \lambda$ and that $p \geq 1/2$. Show that X doesn't explode and that X admits no invariant probability measure.

For the same reasons X doesn't explode. Let μ be an invariant measure of X. The measure μ is supported on $\mathbb{N}$, so it must be σ-finite. Therefore, since for every $t \geq 0$, $\mu P(t) = \mu$, we must have $\mu Q = 0$ and thus up to a multiplicative constant we have,

$$\mu_i = \frac{1}{\lambda}\left(\frac{p}{q}\right)^i.$$

Hence, since $p \geq 1/2$, we have $p/q \geq 1$ and the μ has infinite mass. Therefore μ is not a probability measure.

Exercice 2 — *M/M/1 queue invariant measure.*
Consider a shop where customers are served one at a time. Customers arrive at independent times and each arrival time follows an exponential law of parameter $\lambda > 0$. Customers are served one after the other, service times are independent and each service time follows an exponential law of parameter $\mu > 0$. We let X_t denote the number of customers in queue at time $t \geq 0$ (including the customer currently being served). We assume that the queue is empty at time 0 ($X_0 = 0$).

(1) Show that X is a continuous time Markov chain, give its intensity matrix and show that X doesn't blow up.

The process X jump at the minimum time between the arrival of the next client and the servicing of the first client in the queue. Therefore, the jump times of X follow the law of the minimum of two exponential random variables of parameter λ and μ, so is distributed as an exponential variable of parameter $\lambda + \mu$. The associated jump process can only go to $i+1$ and $i-1$ when in state $i \geq 1$, the probability of jumping from i to $i+1$ is $\mathbb{P}(Exp(\mu) \geq Exp(\lambda)) = \frac{\lambda}{\lambda+\mu}$. Therefore, X is a continuous time Markov chain with intensity matrix

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & & & \\ \mu & -\lambda-\mu & \lambda & 0 & & \\ 0 & \mu & -\lambda-\mu & \lambda & 0 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The diagonal coefficients of Q are lower bounded, so X doesn't explode.

(2) Show that the process X admits a reversible measure.

If μ is a reversible measure then $\mu_i Q_{ij} = \mu_j Q_{ji}$. Taking $j = i+1$, we obtain $\mu_{i+1}\mu = \mu_i \lambda$ And we find $\mu_i = \mu_0 \rho^i$. Conversely we can check that this is an invariant measure.

(3) Using Exercise 1, give a necessary and sufficient condition for X to admit an invariant probability measure π. Express π in terms of $\rho = \lambda/\mu$. This corresponds to the set up of exercise 1, with $q(i) = \lambda + \mu$ and $p = \frac{\lambda}{\lambda+\mu}$. From exercise 1 we know that the process admits an invariant probability distribution if and only if $p < 1/2$,

this corresponds to $\rho < 1$. We have,

$$\pi_i = (1-\rho)\rho^i.$$

(4) Assume that the condition of question (2) is fufilled, on average how much time do we have to wait until the first tile we see 0 customers in the queue ? In the large t limit, what is the probability that here is no customers left in the queue ?
We know that as $t \to \infty$ X_t converges in law to the π and $\pi_i = \frac{1}{q(i)\,\mathbb{E}_i[T^i]}$. We have $\mathbb{E}_0[T^0] = \frac{1}{q(0)\pi_0} = \frac{\mu(\lambda+\mu)}{\mu-\lambda}$. $\lim_{t\to\infty} \mathbb{P}(X_t = 0) = \pi_0 = 1-\rho$. And the mean of X_t converges to the mean of π which is $\frac{\rho}{1-\rho}$

(5) ($\star$) Find again an invariant measure of X by using the following observation (that you will show). Let π be an invariant measure of X, and let $\Pi(s) = \sum_{n\geq 1} \pi(n)s^n$ be its generating function of, then for every $s \in (-1,1)$,

$$\lambda s^2\Pi(s) - (\lambda+\mu)s(\Pi(s) - \pi_0) + \mu(\Pi(s) - \pi_0) - \lambda\pi_0 s = 0.$$

Exercice 3 — *More hitting times.*
Let T^A be the hitting time of A and $h_A(i) = \mathbb{P}_i(T^A < +\infty)$,

(1) Show that the vector $(h_A(i))_{i\in I}$ is the minimal non-negative solution to

$$\begin{cases} h_A(i) = 1 & \text{if } i \in A \\ \sum_{j\in I} q_{i,j} h_A(j) = 0 & \text{otherwise.} \end{cases}$$

(2) Provide a similar interpretation to the minimal nonnegative solution of the system

$$\begin{cases} k(i) = 0, & \text{if } i \in A, \\ \sum_{j\in I} q_{i,j} k(j) = -h_A(i) & \text{otherwise.} \end{cases}$$

(3) Applications: Let Q be the intensity matrix on $I = \{1,2,3,4\}$ given by:

$$Q = \begin{bmatrix} -1 & 1/2 & 1/2 & 0 \\ 1/4 & -1/2 & 0 & 1/4 \\ 1/6 & 0 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For any given initial state, compute the probability of hitting state 3, as well as the expectation of the hitting time of state 4.