TD5 : Kolmogorov's Equation

Exercice 1 — *Birth and death processes.*
We consider the continuous time Markov process X with values in $\mathbb{N}$ and intensity matrix Q given by:

$$q_{i,j} = \begin{cases} \beta_i & \text{si } j = i+1 \\ \delta_i & \text{si } j = i-1 \\ -\beta_i - \delta_i & \text{si } j = i \neq 0 \\ -\beta_i & \text{si } j = i = 0 \\ 0 & \text{sinon,} \end{cases}$$

where the β_i et δ_i are assumed to be nonnegative.

(1) Let $i \in \mathbb{N}$, using Kolmogorov's backward equation, write down a system of differential equations satisfied by $p_{ij}(t) = \mathbb{P}_i(X_t = j)$. Kolmogorov's equation states that $P'(t) = P(t)Q$ so given $i \in \mathbb{N}$ we have,

$$\begin{cases} p'_{i0}(t) = -\beta_0 p_{i0}(t) + \delta_1 p_{i1}(t) \\ p'_{ij}(t) = \beta_{j-1} p_{i,j-1}(t) - (\beta_j + \delta_j) p_{i,j}(t) + \delta_{j+1} p_{i,j+1}(t) \text{ for every } j \geq 1. \end{cases}$$

(2) Assume that for all $i \in \mathbb{N}$, $\delta_i = 0$, show that the system of differential equations of the previous question admits at most one solution on $\mathbb{R}_+$.
We show that there is a unique solution by induction on j. If p, q both solve the system of the previous question, then the $j = 1$ equation imposes for all i, $p_{i0} = q_{i0}$. Let $J \geq 1$ if for all $i \geq 1$ and $j \leq J$ we have $p_{ij} = q_{ij}$, then the $j = J+1$ equation of the system imposes $p_{i,J+1} = q_{i,J+1}$, thus the result.

(3) Assume that for all $i \in I$, $\beta_i = \beta$ and $\delta_i = 0$, show that when the process is started at $X_0 = 0$, the law of X_t is Poisson with parameter βt. Let $p_j(t) = e^{-\beta t}\frac{(\beta t)^j}{j!}$, we have $p'_0(t) = -\beta p_0(t)$ and for $j \geq 1$, $p_j(t) = -\beta p_j(t) + \beta p_{j-1}(t)$. So according to question (1), $\mathbb{P}(X_t = j)$ and $p_j(t)$ solves the same differential system, so according to question (2) they are equal.

(4) Assume that for all $i \in I$, $\beta_i = 0$ and $\delta_i = i\delta$, show that when the process is started at $X_0 = C > 0$, the law of X_t is binomial with parameters $(C, e^{-\delta t})$. We are going to proceed as in the previous question. Note that in this case we have not proven uniqueness of solutions to the system. But, because the birth rate is zero ($\beta_i = 0$) we have $X_t \leq C$ $\mathbb{P}_C$ almost surely. Therefore $\mathbb{P}_C(X_t = j) = 0$ for $j > C$ and since $(p_{ij})_{1 \leq j \leq C}$ is a finite dimensional vector it is characterized by the ODE it satisfies. Thus, it suffices to show that $p_j(t) = \binom{C}{j} e^{-\delta t j}(1 - e^{-\delta t})^{C-j}$ satisfies the right system of differential equations. First all of it is clear that $p_j = 0$ for $j > C$. In addition,

for $j \leq C$ we have,

$$p_j'(t) = \binom{C}{j}(-\delta j)e^{-\delta tj}(1-e^{-\delta t})^{C-j} + \binom{C}{j}e^{-\delta tj}(1-e^{-\delta t})^{C-j-1}(C-j)(-\delta)(-e^{-\delta t}).$$

Since $(C-j)\binom{C}{j} = (j+1)\binom{C}{j+1}$ it follows that

$$p_j'(t) = -\delta_j p_j(t) + \delta_{j+1}p_{j+1}(t).$$

Thus the result.

(5) Give an interpretation of the Markov chain in (4), and use it to compute the value of the extinction probability $\mathbb{P}_C(X_t = 0)$.

This corresponds to a model started with C independent bacterias, each of them dies according to an exponential time of parameter δ and X_t is the number of bacterias at time t. In particular, the probability that a chosen bacteria is alive at time t is $\mathbb{P}(Exp(\delta) \leq t) = 1 - e^{-\delta t}$. Since there are C independent bacterias the probability that they are all dead at time t is $(1-e^{-\delta t})^C$.

Exercice 2 — *Kolmogorov Equations Makes Life Easier.*
Let I be a set and X a continuous time Markov chain with intensity matrix Q. Let λ be a signed measure on I and $f : I \to \mathbb{R}$, we let $g_\lambda(t) = \mathbb{E}_\lambda[f(X_t)] := \sum \lambda(\{i\})\,\mathbb{E}_i[f(X_t)]$. In this exercise, we assume that the integrals/sums are well-defined and that we can derive under the integral/sum. Note this is in particular always the case when I is finite, but could be false in general.

(1) Identifying the measure λ with the lign vector $(\lambda(\{i\}))_{i\in I}$ and the function f with the column vector $(f(j))_{j\in I}$, show we have

$$g_\lambda(t) = \lambda P(t) f.$$

Let $t \geq 0$, since the law X_t under $\mathbb{P}_i$ is $(p_{i,j}(t))_{j\in J}$, we have

$$g_\lambda(t) = \sum_{i\in I} \lambda(i)\mathbb{E}_i[f(X_t)] = \sum_{i,j\in I} \lambda(i)p_{i,j}(t)f(j).$$

(2) Show that g_λ is differentiable and that for every $t \geq 0$,

$$g_\lambda'(t) = \mathbb{E}_{\lambda Q}[f(X_t)] = \mathbb{E}_\lambda[Qf(X_t)].$$

By Kolmogorv's forward and backward equation we have for every $t \geq 0$, $P'(t) = QP(t) = P(t)Q$. So g_λ is differentiable and we have,

$$g_\lambda'(t) = \lambda QP(t)f = \mathbb{E}_{\lambda Q}[f(X_t)]$$
$$g_\lambda'(t) = \lambda P(t)Qf = \mathbb{E}_\lambda[Qf(X_t)].$$

Consider a population of independent bacterias. Each of the bacteria splits into two bacterias after an exponential time of parameter λ. Let X_t denote the number of bacterias in the population at time t. The process X is a Markov chain with intensity matrix,

$$q_{i,j} = \begin{cases} -\lambda i & \text{when } j = i \\ \lambda i & \text{when } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

(3) Let $r \in (-1, 1)$, use the previous questions to find a differential equation satisfied by $g_1(t) = \mathbb{E}_1[r^{X_t}]$.

By the above equation, we have for every $t \geq 0$

$$\begin{aligned} g_1'(t) &= \mathbb{E}_{\delta_1 Q}[r^{X_t}] \\ &= \sum_{j,k} Q_{1k} p_{kj}(t) r^j \\ &= -\lambda \sum_j p_{1j}(t) r^j + \lambda \sum_j p_{2j}(t) r^j \\ &= -\lambda g_1(t) + \lambda g_2(t). \end{aligned}$$

We also recall that the law of X_t started from $X_0 = 2$ is the equal to the law of the sum of two independent copies of X_t started from 1 (Markov property). So $g_2(t) = g_1^2(t)$, and thus g_1 is a solution of

$$\begin{cases} g'(t) = -\lambda g(t) + \lambda g^2(t) \\ g(0) = r. \end{cases}$$

(4) Compute the law of X_t. If we let $f(t) = g_1(t) e^{\lambda t}$, we obtain

$$\begin{cases} f'(t) = \lambda e^{-\lambda t} f^2(t) \\ f(0) = r. \end{cases}$$

This implies is

$$\int_0^t \frac{f'(s)\underset{\cdot}{s}}{f(s)^2} = \int_0^t \lambda e^{-\lambda s} \underset{\cdot}{s}$$

performing the change of variables $x = f(s)$ on the left hand side integral, we obtain

$$\frac{1}{r} - \frac{1}{f(t)} = 1 - e^{-\lambda t}$$

This yields, $g_r(t) = \frac{re^{-\lambda t}}{1 - r(1 - e^{-\lambda t})}$. If X is a geometric random variable of parameter p, we have $\mathbb{E}[r^X] = \frac{rp}{1 - r(1-p)}$, thus X_t is geometric of parameter $e^{-\lambda t}$ and we have,

$$\mathbb{P}(X_t = k) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}.$$

Exercice 3 — *Intensity matrix and transition matrices.*
Let I be a finite set, we set that a matrix P on I is stochastic when all of its entries are nonnegative and for every $i \in I$,

$$\sum_{j \in I} P_{i,j} = 1.$$

Let $Q = (q_{i,j})_{i,j \in I}$ be a matrix on I, for $t \geq 0$, let $P(t) = e^{tQ}$. We aim to show the equivalence of the following three statements:

(i) Q is an intensity matrix
(ii) $P(t)$ is a stochastic matrix for all t in a neighbourhood of 0.
(iii) $P(t)$ is a stochastic matrix for all $t \geq 0$.

(1) Show (ii) and (iii) are equivalent.
Clearly (iii) $\implies$ (ii). Assume (ii), then their exists $\tau > 0$, such that for every $t \in [0, \tau)$ the matrix $P(t)$ is stochastic. Let $t > 0$, let n be an integer strictly larger than τ/t, so that $t/n < \tau$. We have $P(t) = P(t/n)^n$ and the matrix $P(t/n)$ is stochastic, so $P(t)$ is stochastic as the power of a stochastic matrix.

(2) Show (ii) implies (i).
Assume (ii), we have $Q = P'(0)$, so if we let $p_{i,j}(t)$ denote the (i,j) coefficient of $P(t)$, then we have for every $i, j \in I$,

$$Q_{ij} = \lim_{t \to 0} \frac{p_{ij}(t) - \delta_{ij}}{t}.$$

By assumption, their exists $\tau > 0$, such that for every $t \in [0, \tau)$ the matrix $P(t)$ is stochastic in particular, for every $i \in I$,

$$\sum_{j \in I} Q_{ij} = \lim_{t \to 0} \frac{\left(\sum_{j \in I} p_{ij}(t)\right) - 1}{t} = 0.$$

Fix $i, j \in I$, and $t < \tau$, since $P(t)$ is a stochastic matrix, its coefficients belong to $[0,1]$, in particular $p_{ii}(t) - 1 \leq 0$, so $Q_{ii} \leq 0$ and $p_{ij}(t) \geq 0$ so $Q_{ij} \geq 0$.

(3) We now suppose (i) is satisfied.
(a) Show that for all i and all t, we have $\sum_j P(t)_{i,j} = 1$. (Hint: Use the ODE satisfied by P)
Let $\mathbf{1} \in \mathbb{R}^I$ be the vector with every coordinates equal to 1 and for every $t \geq 0$, let $v(t) = P(t)\mathbf{1}$. The vector v solves,

$$\begin{cases} v'(t) = Qv(t) \text{ on } \mathbb{R}_+^* \\ v(0) = \mathbf{1}. \end{cases}$$

The rows of Q sum to 0. this means that for every $Q\mathbf{1} = 0$, so $w(t) = \mathbf{1}$ solve the same equation as v. By uniqueness, we have $v = w$ and for every $t \geq 0$, $P(t)\mathbf{1} = \mathbf{1}$.

(b) Show the entries of the matrix $P(t)$ are nonnegative. Let X be the Markov chain with intensity matrix Q, we have for every $t \geq 0$,

$$p_{i,j}(t) = \mathbb{P}(X_t = j | X_0 = i) \geq 0.$$

So the matrix $P(t)$ is stochastic.