TD4 : Stopping times and Continuous time Markov Chains

Exercice 1 — *Elementary results on stopping times.*
Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. We denote by $\mathbb{F}_+ = (\mathcal{F}_{t+})_{t\geq 0}$ the right-continuous completion of the filtration $\mathbb{F}$, where for every $t \geq 0$

$$\mathcal{F}_{t+} = \bigcap_{t<s} \mathcal{F}_s.$$

(1) Let T be an $\mathbb{F}$-stopping time, let $t \geq 0$ show that $\{T < t\}$ and $\{T = t\}$ are $\mathcal{F}_t$-measurable.

Let $t \geq 0$, we have $\{T < t\} = \cup_{n\geq 1}\{T \leq t(1 - \frac{1}{n})\}$. For every $n \geq 1$, $\{T \leq t(1 - \frac{1}{n})\} \in \mathcal{F}_{t(1-\frac{1}{n})} \subset \mathcal{F}_t$. So, $\{T < t\}$ is $\mathcal{F}_t$-measurable. Then, the set $\{T = t\} = \{T \leq t\} - \{T < t\}$ is also $\mathcal{F}_t$-measurable.

(2) Let T be a random variable valued in $\mathbb{R}_+ \cup \{+\infty\}$, show that T is a $\mathbb{F}_+$-stopping time if and only if for every $t \geq 0$, $\{T < t\} \in \mathcal{F}_t$.

Assume that T is a $\mathbb{F}_+$-stopping time. Let $t \geq 0$, we have $\{T < t\} = \cup_{n\geq 1}\{T \leq t - \frac{1}{n}\}$, for every $n \geq 1$, $\{T \leq t - \frac{1}{n}\} \in \cup_{t-1/n<s}\mathcal{F}_s \subset \mathcal{F}_{t+1/n-1/n} = \mathcal{F}_t$. So, $\{T < t\}$ is $\mathcal{F}_t$-measurable. Conversely, assume that for every $t \geq 0$, $\{T < t\}$ is $\mathcal{F}_t$-measurable. Let $s > t$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $t + 1/n \leq s$. For every $n \geq N$, $\{T < t + 1/n\} \in \mathcal{F}_{t+1/n} \subset \mathcal{F}_s$. And,

$$\{T \leq t\} = \cap_{n\geq N}\{T < t + 1/n\} \in \mathcal{F}_s.$$

Thus, $\{T \leq t\} \in \mathcal{F}_{t+}$.

(3) Let T, S be two $\mathbb{F}$-stopping times, assume that $T \leq S$ almost surely and show that $\mathcal{F}_T \subset \mathcal{F}_S$. Let $A \in \mathcal{F}_T$, and let $t \geq 0$. Since $T \leq S$, $\{S \leq t\} \subset \{T \leq t\}$, so

$$A \cap \{S \leq t\} = (A \cap \{T \leq t\}) \cap \{S \leq t\} \in \mathcal{F}_t.$$

This shows that $A \in \mathcal{F}_S$ and $\mathcal{F}_T \subset \mathcal{F}_S$.

(4) Let T be an $\mathbb{F}$-stopping time, for every $n \geq 1$ show that $T_n = \frac{\lceil 2^n T \rceil}{2^n}$ is a $\mathbb{F}$-stopping time and that almost surely $(T_n)_n$ is a decreasing sequence that converges to T.

Let $n \geq 1$, the key observation is that T_n takes discrete values so $\{T_n \leq t\} = \cup_{k\leq t2^n}\{T_n = k/2^n\}$. Now let $k \leq t2^n$ be an integer, we have

$$\{T_n = k/2^n\} = \{\lceil 2^n T\rceil = k\} = \{(k-1)/2^n < T \leq k/2^n\} = \{T \leq k/2^n\} - \{T \leq (k-1)/2^n\}.$$

So, $\{T_n = k/2^n\} \in \mathcal{F}_t$ and T_n is a stopping time.

Let E be a metric space equipped with its Borel σ-algebra $\mathcal{E}$. Let $A \in \mathcal{E}$ be a measurable set and let X be an E valued $\mathbb{F}$-adapted process. Define,

$$T_A = \inf\{t \geq 0, X_t \in A\}.$$

(5) Assume that X has right continuous trajectories. Let $O \subset E$ be an open set, show that T_O is a $\mathbb{F}_+$-stopping time. Let $t \geq 0$, if we show that $\{T_O < t\} = \cup_{s<t, s \in \mathbb{Q}_+} \{X_s \in O\}$ we are done. Inclusion from right to left is clear. Let $\omega \in \Omega$ such that for every $T_O(\omega) < t$. By contradiction, assume that for every $s < t$, $X_s(\omega) \notin O$. Then, since O^c is closed and $\mathbb{Q}_+ \cap [0,t)$ is dense to the right in $[0,t)$ we would have for every $s \in [0,t), X_s(\omega) \notin O$, which would imply $T_0 \geq t$, this is absurd.

(6) Assume that X has continuous trajectories. Let $F \subset E$ be a closed set, show that T_F is a $\mathbb{F}$-stopping time. Let $Y_t = d(F, X_t)$, Y is adapted and has continuous trajectories, furthermore $T_F = \inf\{t \geq 0, Y_t = 0\}$. For every $t \geq 0$;

$$
\begin{aligned}
\{T_F \leq t\} &= \cup_{s \leq t} \{X_s \in F\} \\
&= \cup_{s \leq t} \{Y_s = 0\} \\
&= \{\inf_{s \leq t} Y_s = 0\} \\
&= \{\inf_{s \in \mathbb{Q}_+ \cap [0,t]} Y_s = 0\} \\
&\in \mathcal{F}_t.
\end{aligned}
$$

Note that the first equality is true because $X_{T_F} \in F$ by continuity of the trajectories and closeness of F.

(7) Give an example where T_O is not a $\mathbb{F}$-stopping time. Let B be a ± 1 Bernoulli random variable with parameter $p \in (0,1)$, define $X_t = tB$, let $\mathbb{F}$ be the natural filtration associated to X. Then, $\mathcal{F}_0$ is the trivial filtration and for $t > 0$, $\mathcal{F}_t = \sigma(B)$. Let T be the hitting time of $\mathbb{R}_+^*$ for the process X. We have, $\{T \leq 0\} = \{B = 1\} \notin \mathcal{F}_0$ so T is not a $\mathbb{F}$-stopping time.

Exercice 2 — *Transition Matrix Computations.*
Let X be a continuous time Markov chain valued in a discrete set I. For every $i, j \in I$ and $t \geq 0$ we define $p_{i,j}(t) = \mathbb{P}(X_t = i | X_0 = j)$ and we let $P(t) = (p_{i,j}(t))_{i,j \in I}$.

(1) Recall the relation between the intensity matrix Q of the process X and $(P(t))_{t \geq 0}$. We have, for every $t \geq 0$, $P(t) = e^{tQ}$.

(2) Let $n \geq 1$ and $Q \in M_n(\mathbb{R})$ be a diagonalizable matrix, let $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ be the eigenvalues of Q. Show that there exists a unique d-uplet $A_1, \ldots, A_d \in M_n(\mathbb{R})$ such that for every $t \geq 0$,

$$
e^{tQ} = \sum_{i=1}^{d} e^{\lambda_i t} A_i.
$$

Show that this result becomes false when Q is not assumed to be diagonalizable. (Hint : $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.)

For every $i, j \in \{1, \ldots, n\}$, let E_{ij} denote the matrix whose coefficients are all 0 except the coefficient (i,j) which is equal to 1. Let $D \in M_n(\mathbb{R})$ be a diagonal matrix, let $\lambda_1, \ldots, \lambda_d$ its eigenvalues, for every $i \in \{1, \ldots, d\}$ let $N_i = \{k \in$

$\{1,\ldots,n\}, D_{kk}=\lambda_i\}$ and $n_i=\#N_i$. We have, for every $t\geq 0$,

$$e^{tD}=\sum_{i=1}^{d}e^{\lambda_i t}\sum_{k\in N_i}E_{kk}.$$

Now let $Q\in M_n(\mathbb{R})$ be a diagonalizable matrix, let $P\in M_n(\mathbb{R})$ be an invertible matrix and $D\in M_n(\mathbb{R})$ be a diagonal matrix such that $Q=PDP^{-1}$. The matrix D admits a decomposition as above. For every $i\in\{1,\ldots,d\}$ let $A_i=\sum_{k\in N_i}PE_{kk}P^{-1}$. We have for every $t\geq 0$,

$$e^{tQ}=Pe^{tD}P^{-1}=\sum_{i=1}^{d}e^{\lambda_i t}A_i.$$

For the counter-example, the proposed matrix satisfies $e^{tQ}=e^t\begin{pmatrix}1 & t\\ 0 & 1.\end{pmatrix}$

(3) Compute $P(t)$ for every $t\geq 0$ assuming that $I=\{1,2\}$ and that,

$$Q=\begin{pmatrix}-\lambda & \lambda\\ \mu & -\mu\end{pmatrix}.$$

If $(\lambda,\mu)=(0,0)$ the question is trivial, we assume this is not the case. The eigenvalues of Q are $\lambda_1=0$ and $\lambda_2=-(\lambda+\mu)$ with eigenvectors $v_1=(1,1)$ and $v_2=(\lambda,-\mu)$. Therefore, there exists $A,B\in M_2(\mathbb{R})$ such that for every $t\geq 0$, $e^{tQ}=A+e^{-(\lambda+\mu)t}B$. By evaluating $t\mapsto e^{tQ}$ and its derivative at $t=0$, we obtain

$$\begin{cases}A+\quad B=I_2\\ \qquad\lambda_2 B=Q.\end{cases}$$

Thus, $e^{tQ}=I_2+\frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu}Q$.

(4) Compute $P(t)$ for every $t\geq 0$ assuming that $I=\{1,2,3\}$ and that,

$$Q=\begin{pmatrix}-2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -3\end{pmatrix}.$$

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Exercice 3 — *Explosion time.*
Let X be a continuous time Markov chain, with intensity matrix Q such that for every $i\in I$, $q(i)\neq 0$. Let $(Y_n)_n$ denote the jump process of X and ζ be the explosion time of X. Let $(E_n)_{n\geq 1}$ be independent exponential random variables with parameters $\lambda_n\in(0,+\infty)$. We let E denote the value in $\mathbb{R}_+\cup\{+\infty\}$ of $\sum_{n\geq 1}E_n$.

(1) Assume that $\sum_{n\geq 1}\frac{1}{\lambda_n}<+\infty$, show that $\mathbb{P}(E<+\infty)=1$.
Assume by contradiction that $\mathbb{P}(E=\infty)>0$, then $\mathbb{E}E=+\infty$. But, $\mathbb{E}E=\sum_{n\geq 1}\mathbb{E}E_n=\sum_{n\geq 1}\frac{1}{\lambda_n}<+\infty$. So, E is almost surely finite.

(2) Assume that $\sum_{n\geq 1}\frac{1}{\lambda_n}=+\infty$, show that $\mathbb{P}(E=+\infty)=1$.

Consider the random variable e^{-E}, let us show that $\mathbb{P}(e^{-E}=0)=1$. For every $N\geq 1$, let $J_N=\sum_{n=1}^{N}J_n$, by definition, almost surely we have $E=\lim_{N\to\infty}J_N$. And by dominated convergence,

$$\mathbb{E}e^{-E}=\lim_{N\to\infty}\mathbb{E}e^{-J_N}.$$

We have,

$$\mathbb{E}e^{-J_N}=\prod_{n=1}^{N}\left(1+\frac{1}{\lambda_n}\right)^{-1}=\exp\left(-\sum_{n=1}^{N}\log\left(1+\frac{1}{\lambda_n}\right)\right).$$

The sum $\sum_{n\geq 1}\log\left(1+\frac{1}{\lambda_n}\right)$ has the same nature as $\sum_{n\geq 1}\frac{1}{\lambda_n}$. Indeed, if $\lambda_n\to 0$ then $\log\left(1+\frac{1}{\lambda_n}\right)\sim\lambda_n$ and otherwise both of the sum diverge. So, $\sum_{n=1}^{\infty}\log\left(1+\frac{1}{\lambda_n}\right)=+\infty$, hence $\mathbb{E}e^{-E}=\lim_{N\to\infty}\mathbb{E}e^{-J_N}=0$. The random variable e^{-E} is nonnegative and as 0 mean, therefore, almost surely $e^{-E}=0$. It follows that $\mathbb{P}(E=+\infty)=1$.

(3) We say that X is a Yule process when, X is $\mathbb{N}$-valued and for every $n\in\mathbb{N}$, $Y_n=Y_0+n$. Let λ be a probability measure on $\mathbb{N}$, assume that X is a Yule process and compute the probability of explosion of X under $\mathbb{P}_\lambda$ in terms of $(q_n)_n$.

When X is a Yule process, conditionally on Y_0 the jump times $(S_n)_n$ of X are independent exponential random variables with parameters $q(Y_0+n)$ and $\zeta=\sum_{n\geq 1}S_n$. So, conditionally on Y_0, ζ is almost surely finite or almost surely infinite, depending on the nature of $\sum_n\frac{1}{q(n+Y_0)}$. So, since $\sum_n\frac{1}{q(n+Y_0)}$ as the same nature as $\sum_n\frac{1}{q(n)}$, we deduce that,

$$\mathbb{P}(\zeta<+\infty)=\begin{cases}1 \text{ if } \sum_n\frac{1}{q(n)}<+\infty\\ 0 \text{ otherwise }.\end{cases}$$

(4) Show that X almost surely doesn't blow up if and only if $\mathbb{P}\left(\sum_n\frac{1}{q(Y_n)}=+\infty\right)=1$.

Same idea as the previous question, conditionally on $(Y_n)_n$ the jump times are independent exponential random variables with parameter $(q(Y_n))_n$. So, conditionally on $(Y_n)_n$ the blow up time ζ of X is a sum of independent exponential random variables with parameters $(q(Y_n))_n$. This sum is infinite if and only if $\sum_n\frac{1}{q(Y_n)}=+\infty$, therefore,

$$\mathbb{P}(\zeta=+\infty)=\mathbb{P}\left(\sum_n\frac{1}{q(Y_n)}=+\infty\right).$$

(5) Show that in the following cases the condition of question (4) is satisfied and so X almost surely doesn't blow up.

(a) The state space is finite.

(b) $\sup_{i\in I}q(i)<+\infty$.

(c) The jump process Y admits a recurrent state. That is, there exists $i \in I$ such that almost surely the set $\{n \in \mathbb{N}, Y_n = i\}$ is infinite.

(a) When I is finite, condition (b) and (c) are satisfied, so it is enough to show that X doesn't blow up under (b) or (c).

(b) Assume that $q = \sup_{i \in I} q(i) < +\infty$, then $\sum_{n=1}^{N} \frac{1}{q(Y_n)} \geq N/q$, so

$$\mathbb{P}\left(\sum_n \frac{1}{q(Y_n)} = +\infty\right) = 1.$$

(c) Let $i \in I$, assume that i is a recurrent state of Y and let $q = q(i)$. Let $K \in \mathbb{N}$, almost surely, Y visits state i at least K times so $\mathbb{P}\left(\sum_n \frac{1}{q(Y_n)} \geq K\right) = 1$. And,

$$\mathbb{P}\left(\sum_n \frac{1}{q(Y_n)} = +\infty\right) \geq \mathbb{P}\left(\forall K \in \mathbb{N}, \sum_n \frac{1}{q(Y_n)} \geq K\right) = 1.$$