ENS de Lyon — Mathematic department
Stochastic processes

Master 1 — Spring 2025
V. Issa & E. Jacob

TD3 : Transition semigroups and more on Poisson processes

Exercice 1 — *Transition semigroups.*
In this exercise, questions are independent.

(1) Let $d \geq 1$, let $(\mu_t)_{t\geq}$ be a family of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Assume that there exists a measurable function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that for every $t \geq 0$, the characteristic function of μ_t is given by $\xi \mapsto e^{t\phi(\xi)}$. Let $(X_t)_{t\geq 0}$ be independent random variables such that for every $t \geq 0$, X_t has law μ_t. For every $t \geq 0$ and $x \in \mathbb{R}^d$, let $P_t(x, \cdot)$ be the law of $x + X_t$. Show that $(P_t)_{t\geq 0}$ is a Markov semigroup on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Let $t, s \geq 0$ and $x \in \mathbb{R}^d$, we are going to show that the characteristic functions of $P_{t+s}(x, \cdot)$ and $\int_E P_s(x, dy)P_t(y, dx')$ are equal. Let $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\int_E \exp(ix' \cdot \xi) P_{t+s}(x, dx') &= \mathbb{E}\left[\exp(i\xi \cdot (x + X_{t+s}))\right] \\
&= \exp(\xi \cdot x + (t+s)\phi(\xi) \\
&= \mathbb{E}\left[\exp(i\xi \cdot (x + X_s + X_t)\right] \\
&= \mathbb{E}\left[\mathbb{E}[\exp(i\xi \cdot (x + X_s + X_t)|X_s]\right] \\
&= \mathbb{E}\left[\int_E \exp(i\xi \cdot x') P_t(x + X_s, dx')\right] \\
&= \int_E \int_E \exp(i\xi \cdot x') P_t(y, dx') P_s(x, dy) \\
&= \int_{x' \in E} \exp(i\xi \cdot x') \int_{y \in E} P_t(y, dx') P_s(x, dy).
\end{aligned}$$

(2) Using the previous question, show that the Poisson semigroup, the Gaussian semigroup, and the Cauchy semigroup from lecture 2 are indeed Markov semigroups. Briefly explain why it's not possible to use question (1) to show that the lognormal semigroup is indeed a Markov semigroup.

One can compute the characteristic function of each of those semigroups and observe that they are of the form $e^{t\phi(\xi)}$ as in the previous question. With respectively $\phi(\xi) = \lambda(e^{i\xi} - 1)$, $\phi(\xi) = -\xi^2/2$ and $\phi(\xi) = -|\xi|$. For the log normal semigroup there is no explicit expression for the characteristic function and we cannot't proceed as in the previous question.

(3) Let $(E, \mathcal{E})$ be a measurable space, let $(P_t)_{t\geq 0}$ be a Markov semigroup on $(E, \mathcal{E})$ and let $h : E \to \mathbb{R}_+^*$ be a measurable function. For every $t \geq 0$ and every $x \in \mathbb{R}^d$, we

define a measure $\hat{P}_t(x,\cdot)$ on E via,

$$\frac{d\hat{P}_t(x,\cdot)}{dP_t(x,\cdot)} = \frac{1}{h(x)}h(\cdot).$$

Assume that for every $t \geq 0$ and every $x \in \mathbb{R}^d$, we have,

$$h(x) = \int_E h(x')P_t(x,dx').$$

Show that $(\hat{P}_t)_{t\geq 0}$ is a Markov semigroup on $(E,\mathcal{E})$.

Let $t \geq 0$ and $x \in \mathbb{R}^d$, we have

$$\int_E d\hat{P}_t(x,dx') = \frac{1}{h(x)}\int_E h(x')dP_t(x,dx') = 1.$$

In addition for every $A \in \mathcal{E}$, $\hat{P}_t(x,A) = \int_A h(x')dP_t(x,dx') \geq 0$, thus $\hat{P}_t(x,\cdot)$ is a probability measure on E and $x \mapsto \hat{P}_t(x,A)$ is measurable. So $(P_t)_{t\geq 0}$ is a transition kernel on E. Furthermore, for every $t,s \geq 0$ and $x \in E$, we have,

$$\begin{aligned}
\hat{P}_{t+s}(x,dx') &= \frac{h(x')}{h(x)}P_{s+t}(x,dx') \\
&= \frac{h(x')}{h(x)}\int_E P_s(x,dy)P_t(y,dx') \\
&= \int_E \frac{h(y)}{h(x)}P_s(x,dy)\frac{h(x')}{h(y)}P_t(y,dx') \\
&= \int_E \hat{P}_s(x,dy)\hat{P}_t(y,dx').
\end{aligned}$$

In conclusion (P_t) is a Markov semigroup on $(E,\mathcal{E})$.

Exercice 2 — *Thinning property.*
Let $(N_t)_{t\geq 0}$ be a Poisson process of intensity $\lambda > 0$ and $(X_k)_{k\geq 0}$ be iid Bernoulli random variables with parameter $p \in [0,1]$. Let

$$N_t^A = \sum_{k=1}^{N_t} X_k \text{ and } N_t^B = \sum_{k=1}^{N_t}(1-X_k),$$

(1) Let X a Poisson random variable of parameter $\alpha > 0$ and $(B_n)_n$ be iid Bernoulli random variables of parameter p independent from X, show that $Y = \sum_{n=1}^X B_n$ is a Poisson random variable of parameter αp.
Let $N \geq 0$, conditionally on $X = N$, Y is binomial of parameter (N,p). So for every $n \in \{0,\dots,N\}$,

$$\mathbb{P}(Y = n, X = N) = e^{-\alpha}\frac{\alpha^N}{N!}\binom{N}{n}p^n(1-p)^{N-n}.$$

Now fix $n \geq 0$, summing the above display over $N \geq 0$, obtain

$$\mathbb{P}(Y = n) = e^{-\alpha p} \frac{(\alpha p)^n}{n!}.$$

(2) Show that N^A and N^B are independent Poisson processes with intensity λp and $\lambda(1-p)$.
We start by proving that N^A has right continuous trajectories, stationary and independent increments, and that N_t^A is a Poisson distributed random variable with parameter λpt. It is clear that the trajectories of N^A are right continuous and that its increments are stationary and independent (because the increments of N are). Finally according to question 1, N_t^A is N_t^A is a Poisson distributed random variable with parameter λpt. This ensures that N^A is Poisson process with intensity λpt, similarly we can show that N^B is Poisson process with intensity $\lambda(1-p)t$. It remains to prove that N^A and N^B are independent. To do so we can again use the conditionning trick

$$\begin{aligned}
\mathbb{P}(N_t^A = k, N_t^B = l) &= \mathbb{P}(N_t^A = k, N_t^B = l | N_t = k + l)\, \mathbb{P}(N_t = k + l) \\
&= \binom{k+l}{k} p^k (1-p)^l e^{-\lambda} \lambda^{k+l}/(k+l)! \\
&= \frac{e^{-\lambda p}(p\lambda)^k}{k!} \frac{e^{-(1-p)\lambda}((1-p)\lambda)^l}{l!} \\
&= \mathbb{P}(N_t^A = k)\, \mathbb{P}(N_t^B = l).
\end{aligned}$$

(3) A bus station observes arrivals of buses, modeled as a Poisson process of intensity λ. Each arriving bus is either a city bus with probability p, or an intercity bus with probability $1-p$ independently of other buses. What is the law of the time between each city bus ? each intercity bus ? Given $t \geq 0$ what is the expected number of city buses observed at time t given that we have observed $n \in \mathbb{N}$ buses in total ?
The arrivals of city buses correspond to the process $(N_t^A)_{t \geq 0}$, which is a Poisson process with intensity λp. The interarrival times are independent and exponentially distributed with parameter λp. Similarly, the arrivals of intercity buses correspond to the process $(N_t^B)_{t \geq 0}$, thus the interarrival times are independent and exponentially distributed with parameter $\lambda(1-p)$.
According to question 1, $N_t = n$, the number of city buses N_t^A follows a Binomial(n, p) distribution. Thus, the expected number of city buses observed by time t, given that n buses have arrived in total is np.

Exercice 3 — *$M/GI/\infty$ queue.*

Let $X = (X_t)_{t \geq 0}$ be a Poisson process of intensity $\lambda > 0$, we denote $(J_n)_n$ the jump times of X. Let $(Z_n)_n$ be iid random variables, we denote G the cdf of Z_1 and $1/\mu$ the mean of Z_1. Consider the following model, you operate a restaurant in which the n^{th} customer arrives at time J_n and leaves at time $J_n + Z_n$. You want to estimate the number N_t of customers in the shop at time t. Note that for every $t \geq 0$, we have

$$N_t = \sum_n \mathbf{1}\{J_n \leq t \leq J_n + Z_n\}.$$

(1) Let $t \geq 0$, $n \geq 0$ and let U denote a uniform random variable in $[0, t]$, define $p = \mathbb{P}(Z_1 > U)$. Show that conditionally on $X_t = n$, the random variable N_t is Binomial random variable of parameter (n, p).
Let $U_1, \ldots, U_n$ be independent uniform random variables in $[0, t]$ and let σ be the permutation of $\{1, \ldots, n\}$ defined almost surely by $U_{\sigma(1)} < \cdots < U_{\sigma(n)}$. Conditionally on $X_t = n$, $(J_1, \ldots, J_n)$ has the law of $(U_{\sigma(1)}, \ldots, U_{\sigma(n)})$. Thus, still conditionally on $X_t = n$, we have

$$\begin{aligned} N_t &= \sum_{k=1}^n \mathbf{1}\{t \leq J_k + Z_k\} \\ &\overset{(d)}{=} \sum_{k=1}^n \mathbf{1}\{t \leq U_{\sigma(k)} + Z_k\} \\ &\overset{(d)}{=} \sum_{k=1}^n \mathbf{1}\{t \leq U_k + Z_k\}. \end{aligned}$$

So conditionally on $X_t = n$, N_t is a binomial random variable of parameter (n, p) where $p = \mathbb{P}(t \leq Z_1 + U) = \mathbb{P}(t - U \leq Z_1) = \mathbb{P}(U \leq Z_1)$. The last inequality following from the fact that $t - U \overset{(d)}{=} U$.

(2) Let $t > 0$ and $\alpha(t) = \lambda \int_0^t \mathbb{P}(Z_1 > x)dx$, show that $N(t)$ is a Poisson random variable with parameter $\alpha(t)$.
According to the previous given $k \geq 0$ and $n \geq k$, we have

$$\mathbb{P}(N_t = k, X_t = n) = e^{-\lambda t}(\lambda t)^n / n! \binom{n}{k} p^k (1-p)^{n-k}.$$

Summing over $n \geq k$, we obtain

$$\mathbb{P}(N_t = k) = e^{-\lambda t p}(\lambda t p)^k / k!.$$

This means that N_t is a Poisson random variable of parameter $\lambda t p$, finally $p = \mathbb{P}(Z_1 > U) = \frac{1}{t}\int_0^t \mathbb{P}(Z_1 > s)ds$, thus $\lambda t p = \alpha(t)$.

(3) Show that as $t \to \infty$, N_t converges in law toward a Poisson law of parameter $\rho = \lambda/\mu$.
As $t \to \infty$, we have $\alpha(t) \to \lambda \int_0^\infty \mathbb{P}(Z_1 > s)ds = \lambda \mathbb{E}S = \lambda/\mu$. It follows that,

$$\lim_{t \to \infty} \mathbb{P}(N_t = k) = e^{-\rho}\frac{\rho^k}{k!}.$$

In France approximately, 1903896 new cars have been bought each year between 1967 and 2023 (source : CCFA, Comité des Constructeurs Français d'Automobiles).

Assume that the French people buy cars according to a Poisson Process of parameter $\lambda = 1903896$ per year and that there was no car bought before 1967.

(4) Assume that each car owner keeps its car for a duration uniform between 0 and 20 years. What is the expected number of cars in the French fleet in the year 1977 ? what about in the year 1987 ? and Afterward ?

If we keep the notations of the previous section, the expected number of cars in the fleet in the year $1967+t$ is $\alpha(t) = \lambda \int_0^t \mathbb{P}(Z_1 > s)ds$. If Z_1 follows a uniform random variable in $[0, b]$ with $b = 20$, we have for every $s < b$, $\mathbb{P}(Z_1 > s)ds = 1 - s/b$ and $= 0$ otherwise. It follows that α is constant after $t = b$ and for $t \leq b$, we have,

$$\alpha(t) = \lambda \int_0^t 1 - \frac{s}{b} ds = \lambda(t - \frac{t^2}{2b}).$$

We obtain,

$$\alpha(10) = \frac{3}{8}\lambda b \simeq 15M$$
$$\alpha(20) = \lambda b/2 \simeq 20M$$

(5) Answer the previous question now assuming that each owner keeps its car for an exponential duration of parameter $1/10$.

Similar computations yield,

$$\alpha(t) = \lambda \int_0^t \exp(-2s/b)ds = \frac{\lambda b}{2}\left(1 - e^{-2t/b}\right).$$