UNIQUENESS OF PARISI MEASURES FOR ENRICHED CONVEX VECTOR SPIN GLASS

HONG-BIN CHEN AND VICTOR ISSA

Abstract. In the PDE approach to mean-field spin glasses, it has been observed that the free energy of convex spin glass models could be enriched by adding an extra parameter in its definition, and that the thermodynamic limit of the enriched free energy satisfies a partial differential equation. This parameter can be thought of as a matrix-valued path, and the usual free energy is recovered by setting this parameter to be the constant path taking only the value 0. Furthermore, the enriched free energy can be expressed using a variational formula, which is a natural extension of the Parisi formula for the usual free energy.

For models with scalar spins the Parisi formula can be expressed as an optimization problem over a convex set, and it was shown in [2] that this problem has a unique optimizer thanks to a strict convexity property. For models with vector spins, the Parisi formula cannot easily be written as a convex optimization problem.

In this paper, we generalize the uniqueness of Parisi measures proven in [2] to the enriched free energy of models with vector spins when the extra parameter is a strictly increasing path. Our approach relies on a Gateaux differentiability property of the free energy and the envelope theorem.

Keywords and phrases: Mean-field spin glasses, Parisi formula, Parisi measures, Hamilton-Jacobi equations

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1. Introduction

1.1. Preamble. In this paper, we study centered Gaussian processes H_N on $(\mathbb{R}^N)^D$ with the following covariance structure,

$$\mathbb{E}H_N(\sigma)H_N(\tau) = N\xi\left(\frac{\sigma\tau^*}{N}\right), \tag{1.1}$$

where $\xi \in \mathcal{C}^\infty(\mathbb{R}^{D\times D}, \mathbb{R})$ is a fixed smooth function and where $\sigma\tau^*$ denotes the matrix of scalar products

$$\sigma\tau^* = (\sigma_d \cdot \tau_{d'})_{1\leqslant d,d'\leqslant D} \in \mathbb{R}^{D\times D}, \tag{1.2}$$

with $\sigma_d = (\sigma_{di})_{1\leqslant i\leqslant N}$.

We often identify $(\mathbb{R}^N)^D$ with $(\mathbb{R}^D)^N$ and also $\mathbb{R}^{D\times N}$, the set of $D\times N$ matrices, which makes the notation for (1.2) natural. Let S^D, S^D_+, and S^D_{++} denote the set of $D\times D$ symmetric matrices, positive semi-definite $D\times D$ symmetric matrices, and positive definite $D\times D$ symmetric matrices respectively. For $a, b \in S^D_+$, we write $a \geqslant b$ whenever $a - b \in S^D_+$.

Throughout, we will assume that ξ admits a convergent power series and is convex on S^D_+. When $D = 1$, this setup corresponds to the usual setup for mean-field spin glass models with scalar spins as studied in [25]. For $D > 1$, this setup corresponds to mean-field spin glass models with D-dimensional vector spins. Multi-species models [26] and the Potts model [27] are examples of mean-field spin glass models with vector spins.

We give ourselves P_1, a compactly supported probability measure on $\mathbb{R}^D$. Without loss of generality, we may assume that the support is included in the unit ball of $\mathbb{R}^D$. We let $P_N = P_1^{\otimes N}$ denote the law of an element $\sigma \in (\mathbb{R}^D)^N$ with independent rows $\sigma_i = (\sigma_{di})_{1\leqslant d\leqslant D} \in \mathbb{R}^D$ with law P_1. We are interested in the large-N limit of

$$\overline{F}_N(t,0) = -\frac{1}{N}\mathbb{E}\log\int \exp\left(\sqrt{2t}H_N(\sigma) - tN\xi\left(\frac{\sigma\sigma^*}{N}\right)\right)\mathrm{d}P_N(\sigma), \tag{1.3}$$

where $t \geqslant 0$. The term $\xi\left(\frac{\sigma\sigma^*}{N}\right)$ in (1.3) is introduced as a convenience to simplify the expression of the limit; it is constant in classical cases of interest, such as when the coordinates of σ take values in $\{-1, 1\}$ and ξ depends only on the diagonal entries of its argument. In general, the second argument of $\overline{F}_N$ can be any q in the space of nondecreasing functions from $[0, 1)$ to S^D_+, subject to a mild integrability and continuity requirement; the expression in (1.3) is with this argument chosen to be the constant path taking only the value $0 \in S^D_+$. To explain what this space is, let us say that a path $q : [0, 1) \to S^D_+$ is nondecreasing if, for every $u \leqslant v \in [0, 1)$, we have $q(v) - q(u) \in S^D_+$. We then let

$$\mathcal{Q} = \{q : [0,1) \to S^D_+ \mid q \text{ is nondecreasing and càdlàg}\} \tag{1.4}$$

and we set $\mathcal{Q}_p = \mathcal{Q} \cap L^p([0, 1); S^D)$. We postpone the precise definition of $\overline{F}_N(t, q)$ for arbitrary $q \in \mathcal{Q}_2$ to (2.4). In short, this quantity is obtained by adding an energy term in the exponential on the right side of (1.3) to encode the interaction of σ with an external magnetic field, and this external magnetic field has an ultrametric structure whose characteristics are encoded by the path q. Therefore, we call $\overline{F}_N(t,q)$ the **enriched free energy**.

One can check [22, Proposition 3.1] that $\overline{F}_N(0,\cdot)$ does not depend on N. This follows from the fact $P_N = P_1^{\otimes N}$ and that at $t = 0$ the N-body Hamiltonian has the same law as N copies of the 1-body Hamiltonian. For every $q \in \mathcal{Q}_2$, we write

$$\psi(q) = \overline{F}_1(0,q) = \overline{F}_N(0,q). \tag{1.5}$$

When instead P_N is the uniform measure on the sphere of radius $\sqrt{N}$ centered at 0 in $(\mathbb{R}^D)^N$, $\overline{F}_N(0,\cdot)$ depends on N but converges to a differentiable function of q as $N \to +\infty$ [22, Proposition 3.1]. In what follows, we focus on models with $P_N = P_1^{\otimes N}$.

When ξ is convex on S^D_+, the limiting value of $\overline{F}_N(t,0)$ is known, this is the celebrated Parisi formula. The Parisi formula was first conjectured in [29] using a sophisticated non-rigorous argument known as the replica method. The convergence of the free energy as $N \to +\infty$ was rigorously established in [17] in the case of the so-called Sherrington-Kirkpatrick model which corresponds to $D = 1$, $\xi(x) = \beta^2 x^2$ for $\beta > 0$, and $P_1 = \mathrm{Unif}(\{-1,1\})$. The Parisi formula for the Sherrington-Kirkpatrick model was then proven in [16, 31]. This was extended to the case $D = 1$, $P_1 = \mathrm{Unif}(\{-1,1\})$ and $\xi(x) = \sum_{p \geqslant 1} \beta_p^2 x^p$ with $\beta_p \geqslant 0$ in [25]. Some models with $D > 1$ such as multi-species models, the Potts model, and a general class of models with vector spins were treated in [26, 27, 28], under the assumption that ξ is convex on $\mathbb{R}^{D\times D}$. Finally, the case $D > 1$ was treated in general in [10] assuming only that ξ is convex on the set of positive semi-definite matrices. In addition, following [10, 24], the Parisi formula is extended to $\lim_{N\to+\infty} \overline{F}_N(t,q)$ for $q \in \mathcal{Q}_2$. The following version of the Parisi formula is [10, Proposition 8.1]. Throughout, under the assumption that ξ is convex on S^D_+, for every $t \geqslant 0$ and $q \in \mathcal{Q}_2$, we set

$$f(t,q) = \lim_{N\to\infty} \overline{F}_N(t,q). \tag{1.6}$$

We view f as a real-valued function on $[0,+\infty) \times \mathcal{Q}_2$ and we often consider f restricted to a smaller domain. We set

$$L^\infty_{\leqslant 1} = \left\{ p : [0,1) \to S^D \;\middle|\; |p(u)| \leqslant 1 \text{ almost everywhere} \right\}, \tag{1.7}$$

$$\theta(x) = x \cdot \nabla\xi(x) - \xi(x). \tag{1.8}$$

Theorem 1.1 (Generalized Parisi formula [10])**.** *If ξ is convex on S^D_+, then, at every $t \geqslant 0$ and $q \in \mathcal{Q}_2$, the limit of the free energy $\overline{F}_N(t,q)$ is given by*

$$f(t,q) = \sup_{p \in \mathcal{Q} \cap L^\infty_{\leqslant 1}} \left\{ \psi(q + t\nabla\xi \circ p) - t\int_0^1 \theta(p(u))\,\mathrm{d}u \right\}. \tag{1.9}$$

Setting $q = 0$ in (1.9), we recover the usual Parisi formula as presented in [25] for example. Letting $\xi^* : \mathbb{R}^{D\times D} \to \mathbb{R} \cup \{+\infty\}$ denote the convex dual of ξ with respect to the cone S^D_+, that is ξ^* is the function defined by

$$\xi^*(a) = \sup_{b \in S^D_+} \{a \cdot b - \xi(b)\}, \forall a \in \mathbb{R}^{D\times D}, \tag{1.10}$$

the function θ can be rewritten as $\theta(x) = \xi^*(\nabla\xi(x))$ (see Lemma 3.2 below).

Definition 1.2 (Parisi measures)**.** We say that a path $p \in \mathcal{Q} \cap L^\infty_{\leqslant 1}$ is a *Parisi measure* at (t, q) when it is an optimizer in the right-hand side of (1.9).

Usually in the literature, the term Parisi measure refers to the law of $p(U)$ where U is a uniform random variable in $[0, 1)$ and p is an optimizer in the right-hand side of (1.9). Here since we work in the space of paths rather than in the space of probability measures, we choose to use the term Parisi measure to refer to p directly.

Studying the set of Parisi measures is an important question in itself since they are the optimizers in the fundamental Parisi formula. Their study is further motivated by the fact that they describe the limit in law of the overlap matrix under the expected Gibbs measure, that is the limit in law of $\sigma\tau^*/N$ as $N \to +\infty$ where σ and τ are two independent random variables drawn from the expected Gibbs measure.

A first rigorous proof of the uniqueness of Parisi measures at $(t, 0)$ for scalar spins (i.e. $D = 1$) was put forward in [32] but relied on genericity assumptions on ξ. In [32], the function ξ is written $\xi(x) = \sum_p \beta_p^2 x^{2p}$. Assuming that for every p, $\beta_p \neq 0$, the author shows the uniqueness of Parisi measures by differentiating the limit free energy and the Parisi formula with respect to each β_p. In [2], the uniqueness of Parisi measures at $(t, 0)$ is proven in the case of scalar spins (i.e. $D = 1$) without any further assumptions on ξ. In Appendix A we give some key ideas of the proof of the uniqueness of Parisi measures at $(t, 0)$ given in [2] in the context of scalar spins. We also show that thanks to a convexity property from optimal transport, with minimal adaptation, this proof also yields the uniqueness of Parisi measures at (t, q) for every $q \in \mathcal{Q}$ when $D = 1$. In addition, we explain why those arguments do not carry over easily to models with vector spins.

1.2. **Main result.** Let Id be the $D \times D$ identity matrix. Recall that for $a, b \in S^D$ satisfying $a - b \in S^D_+$, we write $a \geqslant b$. We can view $\nabla\xi$ as an $\mathbb{R}^{D\times D}$-valued function. Moreover, we assume that $\nabla\xi(S^D) \subseteq S^D$. Because (1.1) holds, ξ enjoys some extra properties. For example as a consequence of [23, Propositions 6.4 & 6.6], we can assume without loss of generality that ξ satisfies the following monotonicity:

$$\begin{aligned} a, b \in S^D_+, \quad a \geqslant b \quad &\Longrightarrow \quad \xi(a) \geqslant \xi(b). \\ a, b \in S^D_+, \quad a \geqslant b \quad &\Longrightarrow \quad \nabla\xi(a) \geqslant \nabla\xi(b). \end{aligned} \tag{1.11}$$

Throughout, we will also assume that ξ is superlinear on S^D_+, which means

$$\forall M > 0, \quad \exists R > 0 : \quad \inf_{x \in S^D_+,\ |x| \geqslant R} \xi(x)/|x| \geqslant M. \tag{1.12}$$

Also, recall that we assume that ξ is convex on S^D_+ and admits an absolutely convergent power series expansion. These conditions along with the existence of H_N are satisfied by a wide range of interactions. We refer to [10, Section 1.5] for the detail.

For every $a \in S^D$, let $\lambda_{\max}(a)$ and $\lambda_{\min}(a)$ be the largest and smallest eigenvalues of a, respectively. For $a \in S^D_{++}$, we define

$$\mathsf{Ellipt}(a) = \frac{\lambda_{\max}(a)}{\lambda_{\min}(a)}. \tag{1.13}$$

We say that $q \in \mathcal{Q}_\uparrow$ when $q \in \mathcal{Q}_2$, $q(0) = 0$ and there exists a constant $c > 0$ such that for every $u < v$,

$$q(v) - q(u) \geqslant c(v-u)\mathrm{Id} \quad \text{and} \quad \mathsf{Ellipt}(q(v) - q(u)) \leqslant \frac{1}{c}. \tag{1.14}$$

Throughout, we set

$$\mathcal{Q}_{\infty,\uparrow} = \mathcal{Q}_\infty \cap \mathcal{Q}_\uparrow. \tag{1.15}$$

In this paper, we study the uniqueness of Parisi measures at (t,q) for $q \in \mathcal{Q}_2$. We will show the following theorem.

Theorem 1.3 (Uniqueness of Parisi measures)**.** *Assume that ξ is strictly convex on S^D_+ and is superlinear on S^D_+. For every $(t,q) \in (0,+\infty) \times \mathcal{Q}_{\infty,\uparrow}$, there is a unique Parisi measure at (t,q) and it is given by $p = \nabla_q f(t,q)$.*

Here, $\nabla_q f(t,q)$ denotes the Gateaux derivative of $f(t,\cdot)$ at q, see Definition 1.4 below. Note that Theorem 1.3 does not include the case $q = 0$, which is arguably the most interesting. The proof of Theorem 1.3 is in the style of [32] and relies on the Gateaux differentiability of the limit free energy on $(0,+\infty) \times \mathcal{Q}_{\infty,\uparrow}$ as proven in [10, Proposition 8.1].

Definition 1.4 (Gateaux differentiability)**.** Let $(E, |\cdot|_E)$ be a Banach space and denote by $\langle \cdot, \cdot \rangle_E$ the canonical pairing between E and its dual E^*. Let G be a subset of E and, for every $q \in G$, we define

$$\mathrm{Adm}(G,q) = \{\kappa \in E \big| \, \exists r > 0,\ \forall t \in [0,r], q + t\kappa \in G\},$$

to be the set of admissible directions at q. A function $h : G \to \mathbb{R}$ is Gateaux differentiable at $q \in G$ if the following two conditions hold:

(1) For every $\kappa \in \mathrm{Adm}(G,q)$, the following limit exists

$$h'(q,\kappa) = \lim_{t \downarrow 0} \frac{h(q+t\kappa) - h(q)}{t}.$$

(2) There is a unique $y^* \in E^*$ such that for every $\kappa \in \mathrm{Adm}(G,q)$,

$$h'(q,\kappa) = \langle y^*, \kappa \rangle_E.$$

In such a case, we call y^* the Gateaux derivative of h at q and we denote it $\nabla h(q)$.

We close the introduction with a few remarks related to Theorem 1.3. First, as a consequence of the cavity computations performed in [10] the unique Parisi measure encodes the limit law of the overlap.

Remark 1.5 (Limit law of the overlap)**.** *If ξ satisfies the hypotheses of Theorem 1.3, and we further assume that ξ is strictly convex on $\mathbb{R}^{D\times D}$ or that ξ is strongly convex on S^D_+, then it follows from* [10, Proposition 8.8] *and* [10, Theorem 1.4] *respectively, that for every $(t,q) \in (0,+\infty)\times\mathcal{Q}_{\infty,\uparrow}$, the law of the overlap matrix $\sigma\tau^*/N$ under the expected Gibbs measure converges as $N \to +\infty$ to the law of $p(U)$ where p is the unique Parisi measure at (t,q) and U is a uniform random variable in $[0,1)$.*

Furthermore, it follows from [9, Proposition 5.16] that when $q \in \mathcal{Q}_{\infty,\uparrow}$ the unique Parisi measure is strictly increasing.

Remark 1.6 (∞-RSB at $q \in \mathcal{Q}_{\infty,\uparrow}$)**.** *Assume that ξ is as in Theorem 1.3 and that the support of P_1 sans $\mathbb{R}^D$. At every $(t,q) \in (0,+\infty)\times\mathcal{Q}_{\infty,\uparrow}$, the unique Parisi measure p is strictly increasing on $[0,1)$ (i.e. $p(v) \neq p(u)$ for every distinct $v,u \in [0,1)$.*

Finally, since the limit free energy is Lipschitz on $\mathbb{R}_+ \times \mathcal{Q}_1$ [10, Proposition 5.1] and $\mathcal{Q}_{\infty,\uparrow}$ is dense in $\mathcal{Q}_1$, we have the following perturbative result.

Remark 1.7 (Uniqueness up to small perturbation)**.** *There is a constant $C > 0$ such that, for every $t > 0$ and $q \in \mathcal{Q}_1$,*

$$|f(t,0) - f(t,q)| \leqslant C|q|_{L^1}.$$

Therefore, for every $\varepsilon > 0$, we can find $q \in \mathcal{Q}_{\infty,\uparrow}$ such that $f(t,q)$ differs from $f(t,0)$ by ε uniformly in t. Hence, we can roughly state that, in a convex vector spin glass model, up to an arbitrarily small perturbation the Parisi measure is unique.

To conclude, we point out the following technical considerations about multi-species models.

Remark 1.8 (Adaption of results to multi-species models)**.** *We refer to* [8] *for a general setup of a multi-species spin glass model. If the species population ratios are not rational, then a multi-species model is not equivalent to a vector spin glass model as encoded here. So, the results here are not directly applicable. However, they can be adapted straightforwardly using the same arguments and replacing results cited from* [10] *by corresponding results in* [8]*.*

1.3. **Organization of the paper.** In Section 2 we give a precise definition of the enriched free energy. In Section 3 we prove a modified version of the Hopf–Lax formula derived in [10, Corollary 8.2], see (3.2) in Theorem 3.1.

In Section 4 we use this modified Hopf-Lax formula to prove Theorem 1.3. In Section 5, as an application of Theorem 1.3, we upgrade the Gateaux differentiability result [10, Proposition 8.1] to a Fréchet differentiability result, see Definition 5.1 and Theorem 5.2. Finally, in Section 6, we show that Theorem 1.3 implies uniqueness in the critical point representation of [10]. In Appendix A we show the uniqueness of Parisi measures at (t,q) for $q \in \mathcal{Q}$ and $D = 1$ using a different argument relying on the strict convexity property of [2]. We also explain why this argument does not carry over easily when $D > 1$.

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2. Definition of the enriched free energy

The goal of this section is to give a precise definition of the enriched free energy appearing in Theorem 1.1 and Theorem 1.3. We start by giving a definition of $F_N(t,q)$ for a piecewise constant $q \in \mathcal{Q}$. We recall that, for $A, B \in \mathbb{R}^{D\times D}$, we write $A \leqslant B$ when $B - A \in S^D_+$; and $A < B$ when $A \leqslant B$ and $A \neq B$. Let $K \in \mathbb{N}$ and let $q \in \mathcal{Q}$ be a path of the form

$$q = \sum_{k=0}^{K} q_k \mathbf{1}_{[\zeta_k, \zeta_{k+1})} \tag{2.1}$$

where

$$0 = q_{-1} \leqslant q_0 < q_1 < \cdots < q_K, \tag{2.2}$$

$$0 = \zeta_0 < \zeta_1 < \cdots < \zeta_K < \zeta_{K+1} = 1. \tag{2.3}$$

The construction of $F_N(t,q)$ involves a random probability measure with ultrametric properties called the Poisson–Dirichlet cascade. We briefly introduce this object and refer to [25, Section 2.3] for a more detailed explanation. We define

$$\mathcal{A} = \mathbb{N}^0 \cup \mathbb{N}^1 \cup \cdots \cup \mathbb{N}^K$$

with the understanding that $\mathbb{N}^0 = \{\varnothing\}$. We think of $\mathcal{A}$ as a tree rooted at $\varnothing$ such that each vertex of depth $k < K$ has countably many children. For each $k < K$ and $\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k$ the children of α are the vertices of the form

$$\alpha n = (n_1, \ldots, n_k, n) \in \mathbb{N}^{k+1}.$$

The depth of $\alpha = (n_1, \ldots, n_k)$ is denoted by $|\alpha| = k$ and for every $l \leqslant k$, we write

$$\alpha_{|l} = (n_1, \ldots, n_l)$$

to denote the ancestor of α at depth l. Given two leaves $\alpha, \beta \in \mathbb{N}^K$, we denote by $\alpha \wedge \beta$ the generation of the most recent common ancestor of α and β, that is

$$\alpha \wedge \beta = \sup\{k \leqslant K : \alpha_{|k} = \beta_{|k}\}.$$

We attach an independent Poisson process to each non-leaf vertex $\alpha \in \mathcal{A}$ with intensity measure.

$$x^{-1-\zeta_{|\alpha|+1}} \mathrm{d}x.$$

We order increasingly the points of those Poisson processes and denote them by $u_{\alpha 1} \geqslant u_{\alpha 2} \geqslant \dots$. For every $\alpha \in \mathbb{N}^K$, we set $w_\alpha = \prod_{k=1}^K u_{\alpha_{|k}}$ and define

$$v_\alpha = \frac{w_\alpha}{\sum_{\beta \in \mathbb{N}^K} w_\beta}.$$

Definition 2.1 (Poisson–Dirichlet cascade)**.** The Poisson–Dirichlet cascade associated to $(\zeta_k)_{1 \leqslant K+1}$ in (2.3) is the random probability measure on $\mathbb{N}^K$ (the leaves of the tree $\mathcal{A}$) whose weights are given by $(v_\alpha)_{\alpha \in \mathbb{N}^K}$.

Let $(v_\alpha)_{\alpha \in \mathbb{N}^K}$ be the Poisson–Dirichlet cascade associated to $(\zeta_k)_{1 \leqslant K+1}$ in (2.3), chosen to be independent of H_N. Let $(z_\beta)_{\beta \in \mathcal{A}}$ be a family of independent $\mathbb{R}^{D \times N}$-valued Gaussian vectors with independent standard Gaussian entries. We choose $(z_\beta)_{\beta \in \mathcal{A}}$ independent of $(v_\alpha)_{\alpha \in \mathbb{N}^K}$ and H_N. For every $\alpha \in \mathbb{N}^K$, we set

$$w^q(\alpha) = \sum_{k=0}^{K} (q_k - q_{k-1})^{1/2} z_{\alpha_{|k}}$$

with $(q_k)_{0 \leqslant k \leqslant K}$ given in (2.2). The centered Gaussian process $(w^q(\alpha))_{\alpha \in \mathbb{N}^K}$ is $\mathbb{R}^{D \times N}$-valued and has the following covariance structure

$$\mathbb{E}\left[w^q(\alpha) w^q(\alpha')^*\right] = N q_{\alpha \wedge \alpha'}.$$

Henceforth, we write $\mathbb{R}_+ = [0, +\infty)$. For $t \in \mathbb{R}_+$ and q given in (2.1), we define the enriched Hamiltonian

$$H_N^{t,q}(\sigma, \alpha) = \sqrt{2t} H_N(\sigma) - N t \xi\left(\frac{\sigma\sigma^*}{N}\right) + \sqrt{2} w^q(\alpha) \cdot \sigma - \sigma \cdot q_K \sigma,$$

where H_N is given as in (1.1). And we define

$$F_N(t, q) = -\frac{1}{N} \mathbb{E} \log \int \sum_{\alpha \in \mathbb{N}^K} \exp\left(H_N^{t,q}(\sigma, \alpha)\right) v_\alpha \mathrm{d}P_N(\sigma). \tag{2.4}$$

Recall $\mathcal{Q}$ from (1.4) and $\mathcal{Q}_p = \mathcal{Q} \cap L^p([0,1); S^D)$. The expression in the previous display is Lipschitz with respect to (t, q). More precisely for every $t_1, t_2 \in \mathbb{R}_+$ every piecewise constant $q_1, q_2 \in \mathcal{Q}$, as proven in [10, Proposition 3.1], we have

$$|F_N(t_1, q_1) - F_N(t_2, q_2)| \leqslant |q_1 - q_2|_{L^1} + |t_1 - t_2| \sup_{|a| \leqslant 1} |\xi(a)|. \tag{2.5}$$

As a consequence, the free energy admits a unique Lipschitz extension to $\mathbb{R}_+ \times \mathcal{Q}_1$. The relevance of the enriched free energy is encapsulated in Theorem 2.2 below, which is extracted from [10, Corollary 8.7]. In other words, this result states that the enriched free energy is the unique solution of some partial differential equation. This approach was explored in [15], in the replica-symmetric regime, and was later extended to various settings [1, 3, 4, 5, 6, 13, 21, 22, 23, 24].

Recall the notion of Gateaux differentiability in Definition 1.4, the definition of $\mathcal{Q}_{\infty,\uparrow}$ in (1.14) and (1.15), and lastly the limit free energy f in (1.6) (which exists when ξ is convex on S^D_+).

Given a Gateaux differentiable functions $h : \mathcal{Q}_{\infty,\uparrow} \subseteq L^2 \to \mathbb{R}$, we have for $q \in \mathcal{Q}_{\infty,\uparrow}$, $\nabla h(q) \in L^2$. In particular, provided that the integral converges, the following quantity is well-defined $\int_0^1 \xi(\nabla h(q)(u))\mathrm{d}u$, where $u \in [0,1)$ is the variable making $\nabla h(q)$ an element of $L^2 = L^2([0,1))$. Hence, we write for short

$$\int \xi(\nabla h(q)) = \int_0^1 \xi(\nabla h(q)(u))\mathrm{d}u. \tag{2.6}$$

Theorem 2.2 (The free energy solves a PDE [10])**.** *Assume that ξ is convex on S^D_+. The limit free energy f is Gateaux differentiable at every $(t,q) \in (0,+\infty) \times \mathcal{Q}_{\infty,\uparrow}$ and satisfies*

$$\partial_t f - \int \xi(\nabla_q f) = 0 \tag{2.7}$$

everywhere on $(0,+\infty) \times \mathcal{Q}_{\infty,\uparrow}$. Moreover, we have

$$\nabla_q f(t,q) \in \mathcal{Q} \cap L^\infty_{\leqslant 1} \subseteq \mathcal{Q}_2, \qquad \forall (t,q) \in (0,+\infty) \times \mathcal{Q}_{\infty,\uparrow}. \tag{2.8}$$

The quantity $\int \xi(\nabla_q f)$ is understood as in (2.6) and $L^\infty_{\leqslant 1}$ is defined in (1.7).

Recall ψ from (1.5) and thus we have

$$f(0,\cdot) = \psi, \qquad \text{on } \mathcal{Q}_2 \supset \mathcal{Q}_{\infty,\uparrow}. \tag{2.9}$$

According to [11, Theorem 1.1], the Cauchy problem (2.7) on $(0,+\infty) \times \mathcal{Q}_{\infty,\uparrow}$ with initial condition $f(0,\cdot) = \psi$ admits a unique solution (in the viscosity sense). Therefore, Theorem 2.2 can be seen as a characterization of the limit free energy. A variational formula for f analogous to the Parisi formula follows from the Hopf–Lax representation [11, Theorem 1.1 (2)]. It has been conjectured in [22] that Theorem 2.2 is the right way of characterizing the limit free energy for models with nonconvex ξ. In particular, the limit of the usual free energy $\overline{F}_N(t,0)$ (without enrichment) as in (1.3) is conjectured to be the value at $(t,0)$ of the unique solution of (2.7) regardless of the convexity of ξ [22, Conjecture 2.6]. Partial results toward a proof of this conjecture have been obtained in [21, 23, 10, 19]. Because of this, we believe that the enriched free energy is an interesting object for understanding both convex and nonconvex spin glasses and this is why we choose to study it in the remainder of this paper.

3. Modified Hopf–Lax representation of the limit free energy

Henceforth, we always assume that ξ is convex on S^D_+ and we let $f = \lim_{N\to+\infty} F_N$ be the pointwise limit on $\mathbb{R}_+ \times \mathcal{Q}_2$ of the enriched free energy (as in (1.6)). Recall that f is jointly Gateaux differentiable on $(0,+\infty)\times\mathcal{Q}_{\infty,\uparrow}$. We also simplify our notation by writing $\nabla f = \nabla_q f$ for the derivative in the second coordinate.

Our proof of Theorem 1.3 uses the differentiability of the limit free energy to apply the envelope theorem. At the heuristic level, this theorem encapsulates the following phenomenon. Consider $h:\mathbb{R}\to\mathbb{R}$ and $k:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that

$$h(x) = \sup_{x'\in\mathbb{R}} k(x,x'). \tag{3.1}$$

Let $x\in\mathbb{R}$ and let $x'\in\mathbb{R}$ be a maximizer in (3.1). Assume that h is differentiable at x and k is differentiable at (x,x'). Then formally we have,

$$\frac{\mathrm{d}}{\mathrm{d}x}h(x) = \partial_x k(x,x') + \partial_{x'}k(x,x')\frac{\mathrm{d}}{\mathrm{d}x}x',$$

Since x' maximizes $k(x,\cdot)$ we have $\partial_{x'}k(x,x') = 0$ and we are left with

$$\frac{\mathrm{d}}{\mathrm{d}x}h(x) = \partial_x k(x,x').$$

When $\partial_x k(x,\cdot)$ is injective, this computation shows that there is a unique optimizer x'. A rigorous proof of the envelope theorem for functions defined on $\mathbb{R}^d$ can be found in [13, Theorem 2.21].

To apply the envelope theorem to $f(t,q)$, we will not use the Parisi formula (1.9) but a Hopf–Lax type formula (see (3.2) below). A closely related variational formula has already been derived in [10, Corollary 8.2], but it is different in nature. In [10, Corollary 8.2] the dependence in q of the formula is through ψ while in (3.2) it is trough ξ^*.

Theorem 3.1 (Hopf–Lax representation)**.** *Assuming that ξ is strictly convex on S^D_+ and is superlinear on S^D_+, we have that, for every $(t,q)\in(0,+\infty)\times\mathcal{Q}_{\infty,\uparrow}$,*

$$f(t,q) = \sup_{q'\in\mathcal{Q}_\infty}\left\{\psi(q') - t\int \xi^*\left(\frac{q'-q}{t}\right)\right\}. \tag{3.2}$$

Here for $\phi: S^D\to\mathbb{R}$ and $\kappa\in L^2([0,1);S^D)$ we write for short

$$\int \phi(\kappa) = \int_0^1 \phi(\kappa(u))\mathrm{d}u.$$

Before starting the proof of Theorem 3.1, we will adapt two classical results from convex duality to our context in Lemmas 3.2 and 3.3 below. Recall the definition of θ in (1.8).

Lemma 3.2. *Assume that ξ is convex on S^D_+, then we have*

$$\theta(x) = \xi^*(\nabla\xi(x)), \qquad \forall x\in S^D_+.$$

Proof. For every $x \in S^D_+$, it follows from the definition of ξ^* in (1.10) that

$$\begin{aligned}\theta(x) &= \nabla\xi(x)\cdot x - \xi(x)\\ &\leqslant \sup_{x'\in S^D_+}\left\{\nabla\xi(x)\cdot x' - \xi(x')\right\}\\ &= \xi^*(\nabla\xi(x)).\end{aligned}$$

Conversely, for any $x' \in S^D_+$, the convexity of ξ on S^D_+ implies that for every $\lambda \in (0,1]$,

$$\frac{\xi(\lambda x' + (1-\lambda)x) - \xi(x)}{\lambda} \leqslant \xi(x') - \xi(x).$$

Taking $\lambda \to 0$, we get

$$\nabla\xi(x)\cdot(x'-x) \leqslant \xi(x') - \xi(x).$$

Rearranging, we obtain

$$\nabla\xi(x)\cdot x' - \xi(x') \leqslant \nabla\xi(x)\cdot x - \xi(x).$$

Taking the supremum over $x' \in S^D_+$, we get $\xi^*(\nabla\xi(x)) \leqslant \theta(x)$. □

Lemma 3.3. *Assume that ξ is convex on S^D_+ and is superlinear on S^D_+, then ξ^* is locally Lipschitz on $\mathbb{R}^{D\times D}$.*

Proof. For every $y \in \mathbb{R}^{D\times D}$, using the superlinearity of ξ as in (1.12), we have for $x \in S^D_+$ with $|x|$ large enough

$$x\cdot y - \xi(x) \leqslant |x|(|y| - \xi(x)/|x|) \leqslant 0.$$

Thus, for $R > 0$ large enough,

$$\xi^*(y) = \sup_{x\in S^D_+,\ |x|\leqslant R}\{x\cdot y - \xi(x)\} < +\infty$$

Hence ξ^* is finite on $\mathbb{R}^{D\times D}$. Since ξ^* is also convex, we conclude from a classical result [30, Theorem 10.4] that ξ^* is locally Lipschitz. □

Proof of Theorem 3.1. It is clear from Theorem 1.1 and Lemma 3.2 that

$$f(t,q) \leqslant \sup_{q'\in\mathcal{Q}_\infty}\left\{\psi(q') - t\int \xi^*\left(\frac{q'-q}{t}\right)\right\}. \tag{3.3}$$

Let us prove the converse bound. We fix any $q' \in \mathcal{Q}_{\infty,\uparrow}$ and set

$$\gamma_s = q' + \frac{s}{t}(q - q'), \quad \forall s \in [0,t].$$

From the definition of $\mathcal{Q}_{\infty,\uparrow}$ in (1.15), we have $\gamma_s \in \mathcal{Q}_{\infty,\uparrow}$ for every $s \in [0,t]$. According to Theorem 2.2, f is jointly Gateaux differentiable on $(0,+\infty)\times\mathcal{Q}_{\infty,\uparrow}$. Hence, for every $s \in (0,t)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}f(s,\gamma_s) &= \partial_t f(s,\gamma_s) + \langle\dot\gamma_s, \nabla_q f(s,\gamma_s)\rangle_{L^2}\\ &\overset{(2.7)}{=} \int_0^1 \xi(\nabla f(s,\gamma_s))(u)\mathrm{d}u + \langle\dot\gamma_s, \nabla f(s,\gamma_s)\rangle_{L^2}.\end{aligned}$$

By definition of ξ^* in (1.10) and $\nabla f(s,\gamma_s) \in \mathcal{Q}_2$ due to (2.8), we have for every $u \in [0,1)$,

$$\xi^*(-\dot\gamma_s(u)) \geqslant -\dot\gamma_s(u) \cdot \nabla f(s,\gamma_s)(u) - \xi(\nabla f(s,\gamma_s)(u))$$

which along with the previous display implies

$$\frac{\mathrm{d}}{\mathrm{d}s} f(s,\gamma_s) \geqslant -\int_0^1 \xi^*(-\dot\gamma_s(u))\mathrm{d}u = -\int_0^1 \xi^*\left(\frac{q'(u)-q(u)}{t}\right)\mathrm{d}u.$$

Integrating this with respect to s and using $f(0,\gamma_0) = \psi(q')$ due to (2.9), we obtain,

$$f(t,q) \geqslant \psi(q') - t\int_0^1 \xi^*\left(\frac{q'(u)-q(u)}{t}\right)\mathrm{d}u.$$

Since q' is arbitrary, we get

$$f(t,q) \geqslant \sup_{q' \in \mathcal{Q}_{\infty,\uparrow}} \left\{\psi(q') - t\int_0^1 \xi^*\left(\frac{q'(u)-q(u)}{t}\right)\mathrm{d}u\right\}.$$

To conclude, we observe that the set

$$\left\{u \mapsto uc\mathrm{Id} + \sum_{k=1}^K q_k \mathbf{1}_{[\frac{k-1}{K},\frac{k}{K})} \;\middle|\; c > 0;\ q_1 = 0;\ q_2,\dots q_k \in S^D_{++}\right\}$$

is dense in $\mathcal{Q}_\infty$ with respect to L^1-convergence and is contained in $\mathcal{Q}_{\infty,\uparrow}$. Since ξ^* is locally Lipschitz as a consequence of Lemma 3.3 and ψ is continuous in L^1 due to (1.5) and (2.5), by a density argument, we obtain

$$f(t,q) \geqslant \sup_{q' \in \mathcal{Q}_\infty} \left\{\psi(q') - t\int_0^1 \xi^*\left(\frac{q'(u)-q(u)}{t}\right)\mathrm{d}u\right\}.$$

This together with (3.3) completes the proof. □

4. Uniqueness of Parisi measures for vector spins

As discussed in detail at the beginning of Section 3, to prove Theorem 1.3 we use the envelope theorem. To do so, we will again need to adapt a classical result from convex duality for the differentiability of ξ^* (see Lemma 4.1) taking into account that the supremum in the definition of ξ^* in (1.10) is taken over S^D_+ instead of the whole space. We will also need to show that even though $\mathcal{Q}_2$ has an empty interior, $\mathcal{Q}_{\infty,\uparrow}$ satisfies some sort of openness condition. More precisely, we are going to show that given $q \in \mathcal{Q}_{\infty,\uparrow}$ and a Lipschitz function $\kappa : [0,1) \to S^D$, one has $q + \varepsilon\kappa \in \mathcal{Q}_{\infty,\uparrow}$ for every ϵ small enough (see Lemma 4.2).

Lemma 4.1. *Assume that ξ is strictly convex on S^D_+ and is superlinear on S^D_+. Then, ξ^* is continuously differentiable on S^D. In addition, we have*

$$\nabla\xi^*(\nabla\xi(x)) = x, \quad \forall x \in S^D_+.$$

Proof. Since ξ^* is convex, it is enough (e.g. [30, Corollary 25.5.1]) to show that ξ^* is differentiable on S^D to prove that ξ^* is continuously differentiable. We fix any $y \in S^D$ and want to show that ξ^* is differentiable at y. We consider the subdifferential

$$\partial\xi^*(y) = \left\{x \in S^D \;\middle|\; \forall y' \in S^D,\, \xi^*(y') - \xi^*(y) \geqslant x \cdot (y' - y)\right\}. \tag{4.1}$$

According to Lemma 3.3, ξ^* is finite at y. Then, it follows from [14, Proposition 5.3], if $\partial\xi^*(y) = \{x\}$ for some $x \in S^D$, then ξ^* is differentiable at y with $\nabla\xi^*(y) = x$. We are going to show that $\partial\xi^*(y)$ is contained in the set of maximizers of the strictly concave functional, $x \mapsto x \cdot y - \xi(x)$.

Step 1. We show that $\partial\xi^*(y) \subseteq S^D_+$.

Recall (e.g. [18, Theorem 7.5.4] that given $x \in S^D$, we have $x \in S^D_+$ if and only if

$$x \cdot a \geqslant 0, \quad \forall a \in S^D_+.$$

Let $x \in \partial\xi^*(y)$, for every $a \in S^D_+$ we let $y' = y - a$. We have

$$\begin{aligned}\xi^*(y') &= \sup_{x' \in S^D_+} \left\{x' \cdot y' - \xi(x')\right\} \\ &\leqslant \sup_{x' \in S^D_+} \left\{x' \cdot y - \xi(x')\right\} \\ &= \xi^*(y).\end{aligned}$$

Thus,

$$x \cdot a = x \cdot (y - y') \overset{(4.1)}{\geqslant} \xi^*(y) - \xi^*(y') \geqslant 0,$$

which proves that $x \in S^D_+$.

Step 2. We show that $\xi^*(y) \leqslant x \cdot y - \xi(x)$ for every $x \in \partial\xi^*(y)$ and that this implies that ξ^* is differentiable at y.

Fix any $x \in \partial\xi^*(y)$. For every $y' \in S^D_+$, we have

$$x \cdot y - \xi^*(y) \overset{(4.1)}{\geqslant} x \cdot y' - \xi^*(y').$$

According to Step 1, we have $x \in S^D_+$, so by the biconjugation theorem on S^D_+ [12, Theorem 2.2] (for which we need the convexity of ξ and the monotonicity of ξ in (1.11)) we have

$$\xi(x) = \xi^{**}(x) = \sup_{y' \in S^D_+} \left\{x \cdot y' - \xi^*(y')\right\}.$$

Therefore, from the previous two displays, we obtain

$$x \cdot y - \xi^*(y) \geqslant \xi(x),$$

which is the desired inequality. As a consequence, we have $x = x_0$ where x_0 is the unique maximizer of the strictly concave function $x' \mapsto x' \cdot y - \xi(x)$ over S^D_+. Since $x \in \partial\xi^*(y)$ is arbitrary, we get $\partial\xi^*(y) = \{x_0\}$. Hence, as explained in the beginning, we conclude that ξ^* is differentiable at y with $\nabla\xi^*(y) = x_0$.

Step 3. We show that for every $x \in S^D_+$, $\nabla\xi^*(\nabla\xi(x)) = x$.

From Lemma 3.2, we know that

$$\xi^*(\nabla\xi(x)) = x \cdot \nabla\xi(x) - \xi(x).$$

Thus, x is the maximizer of the strictly concave functional $x' \mapsto x' \cdot \nabla\xi(x) - \xi(x')$ over S^D_+. Using the last part of Step 2 with y substituted with $\nabla\xi(x)$, we get $\nabla\xi^*(\nabla\xi(x)) = x$, which completes the proof. □

Recall the definition of $\mathsf{Ellipt}(a)$ in (1.13) and those of $\lambda_{\max}(a)$ and $\lambda_{\min}(a)$ above (1.13). It follows from Weyl's inequalities that $\lambda_{\max}$ and $\lambda_{\min}$ are respectively sub-additive and super-additive. Therefore, given two symmetric matrices $a, b \in S^D$ such that $a + b \in S^D_{++}$, $\lambda_{\max}(a) + \lambda_{\max}(b) \geqslant 0$ and $\lambda_{\min}(a) + \lambda_{\min}(b) > 0$, we have

$$\mathsf{Ellipt}(a+b) \leqslant \frac{\lambda_{\max}(a) + \lambda_{\max}(b)}{\lambda_{\min}(a) + \lambda_{\min}(b)}. \tag{4.2}$$

Lemma 4.2. *Let $\kappa : [0,1) \to S^D$ be a Lipschitz function such that $\kappa(0) = 0$ and $q \in \mathcal{Q}_{\infty,\uparrow}$. We have $q + \varepsilon\kappa \in \mathcal{Q}_{\infty,\uparrow}$ for $\varepsilon > 0$ small enough.*

Proof. Recall that we write $a \geqslant b$ if $a, b \in S^D$ satisfies $a - b \in S^D_+$. Since κ is Lipschitz, there is a constant $L > 0$ such that

$$\kappa(v) - \kappa(u) \leqslant L|u - v|\mathrm{Id}, \quad \forall v, u \in [0,1). \tag{4.3}$$

This implies that, for every $v, u \in [0,1)$, we have

$$\lambda_{\max}(\kappa(v) - \kappa(u)),\ \lambda_{\min}(\kappa(v) - \kappa(u)) \in [-L(v-u), L(v-u)]. \tag{4.4}$$

The definition of $\mathcal{Q}_{\infty,\uparrow}$ in (1.15) and (1.14) gives a constant $c > 0$ such that

$$q(v) - q(u) \geqslant c(v-u)\mathrm{Id} \quad \text{and} \quad \mathsf{Ellipt}(q(v) - q(u)) \leqslant c^{-1}. \tag{4.5}$$

Fixing $u < v$, we have

$$\begin{aligned}(q(v) + \varepsilon\kappa(v)) - (q(u) + \varepsilon\kappa(u)) &\overset{(4.3)(4.5)}{\geqslant} c(v-u)\mathrm{Id} - L\varepsilon(v-u)\mathrm{Id} \\ &= (c - L\varepsilon)(v-u)\mathrm{Id}.\end{aligned}$$

We also have

$$\begin{aligned}
&\mathsf{Ellipt}((q(v)+\varepsilon\kappa(v))-(q(u)+\varepsilon\kappa(u)))\\
&\overset{(4.2)(4.4)}{\leqslant} \frac{\lambda_{\max}(q(v)-q(u))+L\varepsilon|u-v|}{\lambda_{\min}(q(v)-q(u))-L\varepsilon|u-v|}\\
&\overset{(4.5)}{\leqslant} \frac{c^{-1}(\lambda_{\min}(q(v)-q(u))-L\varepsilon|u-v|)+(c^{-1}+1)L\varepsilon|u-v|}{\lambda_{\min}(q(v)-q(u))-L\varepsilon|u-v|}\\
&\leqslant c^{-1}+\frac{(c^{-1}+1)L\varepsilon|u-v|}{\lambda_{\min}(q(v)-q(u))-L\varepsilon|u-v|}\\
&\overset{(4.5)}{\leqslant} c^{-1}+\frac{(c^{-1}+1)L\varepsilon|u-v|}{c|u-v|-L\varepsilon|u-v|}\\
&\leqslant c^{-1}+\frac{(c^{-1}+1)L\varepsilon}{c-L\varepsilon}.
\end{aligned}$$

Furthermore, since $q\in L^\infty$ and κ is Lipschitz, it is clear that $q+\varepsilon\kappa$ is bounded. Since $\kappa(0)=0$ it is also clear that $(q+\varepsilon\kappa)(0)=0$. Now, comparing these and the above two displays with the definitions in (1.14) and (1.15), we conclude that $q+\varepsilon\kappa\in\mathcal{Q}_{\infty,\uparrow}$ for sufficiently small ε. □

Proof of Theorem 1.3. For brevity, we write

$$\mathscr{H}_t(\rho,\rho')=\psi(\rho')-t\int\xi^*\left(\frac{\rho'-\rho}{t}\right),\qquad \forall\rho,\rho'\in\mathcal{Q}_\infty.$$

Then, using this notation and Lemma 3.2, we can rewrite the Parisi formula (1.9) in Theorem 1.1 as

$$f(t,q)=\sup_{p\in\mathcal{Q}\cap L^\infty_{\leqslant 1}}\mathscr{H}_t\left(q,q+t\nabla\xi\circ p\right). \tag{4.6}$$

We fix $(t,q)\in(0,+\infty)\times\mathcal{Q}_{\infty,\uparrow}$. Let $p\in\mathcal{Q}_\infty$ be a Parisi measure (see Definition 1.2) at (t,q). Set $q'=q+t\nabla\xi(p)$. We have $f(t,q)=\mathscr{H}_t(q,q')$, so q' is a maximizer in the right-hand side of (3.2). Let $\kappa:[0,1)\to S^D$ be a Lipschitz function with $\kappa(0)=0$, we have

$$\frac{\mathscr{H}_t(q+\varepsilon\kappa,q')-\mathscr{H}_t(q,q')}{\varepsilon}=-\frac{1}{\varepsilon}\int t\xi^*\left(\frac{q'-q-\varepsilon\kappa}{t}\right)-t\xi^*\left(\frac{q'-q}{t}\right).$$

The integral on the right-hand side is $\int=\int_0^1\mathrm{d}u$ over the parameter of paths. According to Lemma 4.1, the function ξ^* is continuously differentiable on S^D. In particular, ξ^* is locally Lipschitz and we can apply the dominated convergence theorem in the previous display to discover that

$$\lim_{\varepsilon\downarrow 0}\frac{\mathscr{H}_t(q+\varepsilon\kappa,q')-\mathscr{H}_t(q,q')}{\varepsilon}=\left\langle\nabla\xi^*\left(\frac{q'-q}{t}\right),\kappa\right\rangle_{L^2}.$$

By Lemma 4.2, we have $q + \varepsilon\kappa \in \mathcal{Q}_{\infty,\uparrow}$ for every $\varepsilon > 0$ small enough. Due to (4.6), we have $f(t, q + \varepsilon\kappa) \geqslant \mathscr{H}_t(q + \varepsilon\kappa, q')$, therefore

$$\frac{f(t, q + \varepsilon\kappa) - f(t, q)}{\varepsilon} \geqslant \frac{\mathscr{H}_t(q + \varepsilon\kappa, q') - \mathscr{H}_t(q, q')}{\varepsilon}.$$

By Theorem 2.2, $f(t, \cdot)$ is Gateaux differentiable (see Definition 1.4) at q. Letting $\varepsilon \to 0$ in the previous display, we obtain

$$\langle \nabla f(t, q), \kappa \rangle_{L^2} \geqslant \left\langle \nabla \xi^* \left(\frac{q' - q}{t} \right), \kappa \right\rangle_{L^2}.$$

Since κ is an arbitrary Lipschitz function vanishing at $u = 0$ and those functions are dense in L^2, this yields $\nabla f(t, q) = \nabla \xi^* \left(\frac{q'-q}{t} \right)$. In addition, according to Lemma 4.1, we have

$$\nabla \xi^* \left(\frac{q' - q}{t} \right) = \nabla \xi^* (\nabla \xi(p)) = p.$$

So in conclusion, $p = \nabla f(t, q)$. □

5. Fréchet differentiability of the limit free energy

Finally, as an application of Theorem 1.3 we will upgrade the Gateaux differentiability result of [10, Proposition 8.1] to a Fréchet differentiability result.

Definition 5.1 (Fréchet differentiability)**.** Let $(E, |\cdot|_E)$ be a Banach space and denote by $\langle \cdot, \cdot \rangle_E$ the canonical pairing between E and E^*. Let G be a subset of E and $q \in G$, a function $h : G \to \mathbb{R}$ is Fréchet differentiable at $q \in G$ if there exists a unique $y^* \in E^*$ such that the following holds:

$$\lim_{\substack{q' \to q \\ q' \in G}} \frac{h(q') - h(q) - \langle y^*, q' - q \rangle_E}{|q' - q|_E} = 0.$$

In such a case, we call y^* the Fréchet derivative of h at q and we denote it $\nabla h(q)$.

When a function $h : G \to \mathbb{R}$ is Fréchet differentiable at every, $q \in G$ we say that h is Fréchet differentiable everywhere on G.

Theorem 5.2 (Fréchet differentiability of the free energy)**.** *Assume that ξ is convex on S^D_+ and is superlinear on S^D_+. The limit free energy f as in* (1.6) *is Fréchet differentiable everywhere on $(0, +\infty) \times \mathcal{Q}_{\infty,\uparrow}$.*

Recall that from Theorem 2.2 which is extracted from [10] that the Gateaux derivative of f satisfy

$$\partial_t f - \int \xi(\nabla_q f) = 0,$$

everywhere on $(0, +\infty) \times \mathcal{Q}_{\infty,\uparrow}$. Since the Gateaux and Fréchet derivatives coincide when they exist, this partial differential equation can be understood in the Fréchet sense as well.

Proof of Theorem 5.2. In this proof, for clarity, we fix $t \in (0, +\infty)$ and only show that $f(t, \cdot)$ is Fréchet differentiable at every $q \in \mathcal{Q}_{\infty,\uparrow}$. The proof of the joint differentiability is the same but with more cumbersome notations, which we choose to omit here. For $\rho \in \mathcal{Q}_2$ and $\rho' \in \mathcal{Q}_\infty$, we write

$$\mathscr{P}_t(\rho, \rho') = \psi(\rho + t\nabla\xi(\rho')) - t \int \theta(\rho').$$

Before proceeding, we record some properties of $\mathscr{P}_t$. According to [10, Corollary 5.2], ψ is Fréchet differentiable on $\mathcal{Q}_2$. Therefore, $\mathscr{P}_t(\cdot, \rho')$ is Fréchet differentiable at every $\rho \in \mathcal{Q}_2$. We denote by $\nabla_\rho \mathscr{P}_t(\rho, \rho')$ its derivative at ρ, which has the expression:

$$\nabla_\rho \mathscr{P}_t(\rho, \rho') = \nabla\psi(\rho + t\nabla\xi(\rho')).$$

Also, [10, Corollary 5.2] gives that $\nabla\psi$ is Lipschitz in L^2. Using this and the smoothness of ξ, there is a constant $C > 0$ such that

$$\left|\nabla_\rho \mathscr{P}_t(\rho_1, \rho_1') - \nabla_\rho \mathscr{P}_t(\rho_2, \rho_2')\right| \leqslant C\left|\rho_1 - \rho_2\right|_{L^2} + Ct\left|\rho_1' - \rho_2'\right|_{L^2} \tag{5.1}$$

for every $\rho_1, \rho_2 \in \mathcal{Q}_2$ and $\rho_1', \rho_2' \in \mathcal{Q} \cap L^\infty_{\leqslant 1}$. Notice that the boundedness on ρ_1' and ρ_2' is needed here.

According to Theorem 1.1, we have

$$f(t, q) = \sup_{p \in \mathcal{Q}_\infty} \mathscr{P}_t(q, p).$$

Let $(q_n)_n$ be any sequence in $\mathcal{Q}_{\infty,\uparrow} \setminus \{q\}$ such that $q_n \to q$ in L^2. Let $p_n \in \mathcal{Q}_\infty$ be the unique Parisi measure at (t, q_n) given by Theorem 1.3. The theorem also gives $p_n = \nabla f(t, q_n) \in \mathcal{Q} \cap L^\infty_{\leqslant 1}$. By [10, Lemma 3.4], thanks to the monotonicity and boundedness of p_n uniform in n, the sequence $(p_n)_n$ is pre-compact in L^2. Let p be a subsequential limit of $(p_n)_n$. Clearly, we still have $p \in \mathcal{Q} \cap L^\infty_{\leqslant 1}$. Passing to the limit in $f(t, q_n) = \mathscr{P}_t(q_n, p_n)$, we discover that p is a Parisi measure at (t, q). Since Theorem 1.3 ensures that such p is unique, we deduce that the sequence (p_n) converges in L^2 to the unique Parisi measure p at (t, q). It follows that

$$\begin{aligned} f(t, q_n) - f(t, q) &\leqslant \mathscr{P}_t(q_n, p_n) - \mathscr{P}_t(q, p_n) \\ &= \int_0^1 \langle q_n - q, \nabla_\rho \mathscr{P}_t(\lambda q_n + (1-\lambda)q, p_n)\rangle_{L^2} \mathrm{d}\lambda. \end{aligned}$$

We thus have

$$\begin{aligned} &f(t, q_n) - f(t,q) - \langle q_n - q, \nabla_\rho \mathscr{P}_t(q, p)\rangle_{L^2} \\ &\qquad \leqslant \int_0^1 \langle q_n - q, \nabla_\rho \mathscr{P}_t(\lambda q_n + (1-\lambda)q, p_n) - \nabla_\rho \mathscr{P}_t(q, p)\rangle_{L^2} \mathrm{d}\lambda. \end{aligned}$$

Now observe that

$$\begin{aligned} &\int_0^1 \langle q_n - q, \nabla_\rho \mathscr{P}_t(\lambda q_n + (1-\lambda)q, p_n) - \nabla_\rho \mathscr{P}_t(q, p)\rangle_{L^2} \mathrm{d}\lambda \\ &\leqslant |q_n - q|_{L^2} \int_0^1 |\nabla_\rho \mathscr{P}_t(\lambda q_n + (1-\lambda)q, p_n) - \nabla_\rho \mathscr{P}_t(q, p)|_{L^2} \mathrm{d}\lambda. \end{aligned}$$

Due to (5.1), the integral on the right-hand side vanishes as $n \to +\infty$. As a result, we get

$$f(t,q_n) - f(t,q) - \langle q_n - q, \nabla_\rho \mathscr{P}_t(q,p)\rangle_{L^2} \leqslant o(|q_n - q|_{L^2}).$$

In addition, we also have,

$$\begin{aligned} f(t,q_n) - f(t,q) &- \langle q_n - q, \nabla_\rho \mathscr{P}_t(q,p)\rangle_{L^2} \\ &\geqslant \mathscr{P}_t(q_n,p) - \mathscr{P}_t(q,p) - \langle q_n - q, \nabla_\rho \mathscr{P}_t(q,p)\rangle_{L^2} = o(|q_n - q|_{L^2}). \end{aligned}$$

In conclusion,

$$f(t,q_n) = f(t,q) + \langle q_n - q, \nabla_\rho \mathscr{P}_t(q,p)\rangle_{L^2} + o(|q_n - q|_{L^2}).$$

By sequential characterization of the limit, we deduce that f is Fréchet differentiable at q. □

6. Uniqueness in the critical point representation

In [10], a representation of the limit free energy in terms of the critical points of a functional has been proven. More precisely, we consider the functional

$$\mathcal{J}_{t,q}(q',p) = \psi(q') + \langle p, q - q'\rangle_{L^2} + t\int \xi(p). \tag{6.1}$$

We say that $(q',p) \in \mathcal{Q}_2 \times L^2$ is a critical point of $\mathcal{J}_{t,q}$ when

$$\begin{cases} q' = q + t\nabla\xi(p) \\ p = \nabla\psi(q') \end{cases}. \tag{6.2}$$

A consequence of the main result [10, Theorem 1.2] is that, for every $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}_2$ there exists $(q',p) \in \mathcal{Q}_\infty^2$ that is a critical point of $\mathcal{J}_{t,q}$ and such that

$$f(t,q) = \mathcal{J}_{t,q}(q',p).$$

Using Theorem 1.3 we can show that this critical point representation for the free energy is in fact unique as soon as ξ is strictly convex on S_+^D.

Corollary 6.1 (Uniqueness in the critical point representation)**.** *Assume that ξ is strictly convex on S_+^D and is superlinear on S_+^D. Let $(t,q) \in (0,+\infty) \times \mathcal{Q}_{\infty,\uparrow}$, there exists a* unique *critical point $(q',p) \in \mathcal{Q}_\infty$ of $\mathcal{J}_{t,q}$ such that*

$$f(t,q) = \mathcal{J}_{t,q}(q',p).$$

Proof. Let $(q',p) \in \mathcal{Q}_\infty^2$ be a critical point of $\mathcal{J}_{t,q}$. Using $q' - q = t\nabla\xi(p)$, we get

$$\begin{aligned} \mathcal{J}_{t,q}(q',p) &\overset{(6.1)}{=} \psi(q + t\nabla\xi(p)) - t\left(\langle p, \nabla\xi(p)\rangle_{L^2} - \int \xi(p)\right) \\ &\overset{(1.8)}{=} \psi(q + t\nabla\xi(p)) - t\int \theta(p). \end{aligned}$$

If we further assume that $f(t,q) = \mathcal{J}_{t,q}(q',p)$, then we obtain

$$f(t,q) = \psi(q + t\nabla\xi(p)) - t\int \theta(p).$$

This means that p is a Parisi measure, which by Theorem 1.3 imposes that $p = \nabla f(t,q)$. By the critical point condition (6.2), we also have $q' = q + t\nabla\xi(\nabla f(t,q))$, which uniquely characterizes (q',p). □

Appendix A. Uniqueness of Parisi measures for scalar spins

In this section, we expand on the proof of the uniqueness of Parisi measures at $(t,0)$ in the case of scalar spins (i.e. $D=1$) given in [2]. More precisely, using the Kantorovich duality from optimal transport, we show that this proof also yields (still in the case of scalar spins) the uniqueness of Parisi measures at (t,q) for any $q \in \mathcal{Q}_\infty$.

As written, the proof of the strict concavity of the Parisi functional given in [2] assumes that P_1 is the uniform measure on $\{-1,1\}$. We will make this assumption here as well, but it seems that with a bit more care some more general reference measures can be considered using the results of [7, Theorem 4.6].

When $D=1$, we can obtain (3.2) (as done in [11, Proposition A.3]) using an elementary rearrangement argument instead of relying on the Gateaux differentiability of f as in the proof of Theorem 1.3. In this setting, it has been observed that at $q=0$ the Parisi formula can be recast into a strictly concave optimization problem and admits a unique maximizer [2]. Using (3.2), this concavity, and the convexity of the optimal transport cost between probability measures, we show that the uniqueness of Parisi measures at every $q \in \mathcal{Q}_\infty$. Recall the notion of Parisi measures in Definition 1.2.

Proposition A.1 (Uniqueness of Parisi measures for scalar spins). *Assume that $D=1$ and that P_1 is the uniform probability measure on $\{-1,1\}$. For every $(t,q) \in (0,+\infty)\times\mathcal{Q}_\infty$ there is a unique Parisi measure at (t,q).*

Proof. Let $\mathcal{P}(\mathbb{R}_+)$ denote the set of probability measures on $\mathbb{R}_+$. We denote by $\mathcal{P}_2(\mathbb{R}_+)$ (respectively, $\mathcal{P}_\infty(\mathbb{R}_+)$) the set of probability measures on $\mathbb{R}_+$ with finite second moments (respectively, compact supports). We equip $\mathcal{P}_2(\mathbb{R}_+)$ with W_2 the Wasserstein distance defined by

$$W_2(\mu',\mu) = \left(\inf_{\pi\in\Pi(\mu',\mu)} \int |x'-x|^2 \mathrm{d}\pi(x',x)\right)^{1/2}$$

where $\Pi(\mu',\mu)$ denotes the set of probability measures $\pi \in \mathcal{P}_2(\mathbb{R}_+\times\mathbb{R}_+)$ with first marginal μ' and second marginal μ. Let U be a uniform probability measure on $[0,1)$. We consider the map

$$G : \begin{cases} \mathcal{Q}_2 \to \mathcal{P}_2(\mathbb{R}_+) \\ q \mapsto \mathsf{law}(q(U)) \end{cases} .$$

The functional G is an isometric bijection between the metric spaces $(\mathcal{Q}_2, |\cdot|_{L^2})$ and $(\mathcal{P}_2(\mathbb{R}_+), W_2)$. It has been observed in [2, Theorem 2] that the map

$\mu \mapsto \psi(G^{-1}(\mu))$ is strictly concave. More precisely, for every $\mu_0, \mu_1 \in \mathcal{P}_\infty(\mathbb{R}_+)$ and $\lambda \in [0,1]$ we have

$$\psi\left(G^{-1}(\lambda\mu_1 + (1-\lambda)\mu_0)\right) \geqslant \lambda\psi\left(G^{-1}(\mu_1)\right) + (1-\lambda)\psi\left(G^{-1}(\mu_0)\right)$$

with equality if and only if $\lambda = 0$, $\lambda = 1$, or $\mu_0 = \mu_1$.

Given $\mu', \mu \in \mathcal{P}_2(\mathbb{R}_+)$, we denote by $\mathcal{T}_t(\mu', \mu)$ the cost of the optimal transport between μ' and μ with respect to the cost function $(x', x) \mapsto t\xi^*\left(\frac{x'-x}{t}\right)$, that is,

$$\mathcal{T}_t(\mu', \mu) = \inf_{\pi \in \Pi(\mu', \mu)} \int t\xi^*\left(\frac{x'-x}{t}\right) \mathrm{d}\pi(x', x).$$

According to [23, Proposition 2.5], for every $q', q \in \mathcal{Q}_2$ we have

$$\mathcal{T}_t(G(q'), G(q)) = \int_0^1 t\xi^*\left(\frac{q'(u) - q(u)}{t}\right) \mathrm{d}u. \tag{A.1}$$

In other words, the optimal coupling between measures $G(q')$ and $G(q)$ to minimize the aforementioned cost is achieved by the joint $(q'(U), q(U))$ where U is the uniform random variable over $[0,1)$.

Fix any $(t, q) \in (0, +\infty) \times \mathcal{Q}_\infty$ and set $\mu = G(q)$. The formula for f obtained in [11, Proposition A.3] (which is simply (3.2) but with a simpler proof) can be written as

$$f(t, q) = \sup_{\mu' \in \mathcal{P}_\infty(\mathbb{R}_+)} \left\{\psi(G^{-1}(\mu')) - \mathcal{T}_t(\mu', \mu)\right\}.$$

Step 1. We show that the right-hand side in the previous display is a strictly concave optimization problem. In particular, this optimization problem admits a unique maximizer.

The Kantorovich duality theorem (see for example [33, Theorem 5.10]) states that the optimal transport cost $\mathcal{T}_t(\mu', \mu)$ admits the following dual representation

$$\mathcal{T}_t(\mu', \mu) = \sup_{\chi', \chi}\left\{\int \chi' d\mu' - \int \chi d\mu\right\},$$

where the supremum is taken over all pairs of functions $\chi, \chi' : \mathbb{R}_+ \to \mathbb{R}$ such that $\chi'(x') - \chi(x) \leqslant (t\xi)^*(x' - x)$. When the optimal transport cost is written as in the previous display, it is the supremum of a family of linear functions in (μ', μ). In particular, the function $(\mu', \mu) \mapsto \mathcal{T}_t(\mu', \mu)$ is convex on $\mathcal{P}_\infty(\mathbb{R}_+) \times \mathcal{P}_\infty(\mathbb{R}_+)$. Therefore, since the map $\mu' \mapsto \psi(G^{-1}(\mu'))$ is strictly concave according to [2, Theorem 2], we have that for every $\mu \in \mathcal{P}_\infty(\mathbb{R}_+)$, the map $\mu' \mapsto \psi(G^{-1}(\mu')) - \mathcal{T}_t(\mu', \mu)$ is strictly concave on $\mathcal{P}_\infty(\mathbb{R}_+)$.

Step 2. We show that there is a unique Parisi measure at (t, q).

Let p be a Parisi measure at (t, q), let $q' = q + t\nabla\xi(p)$, and let $\mu' = G(q')$. We have

$$\begin{aligned} f(t,q) &\overset{\text{D.1.2}}{=} \psi(q + t\nabla\xi(p)) - t\int \theta(p) \\ &\overset{\text{L.3.2}}{=} \psi(q') - \int_0^1 t\xi^*\left(\frac{q'(u) - q(u)}{t}\right)\mathrm{d}u \\ &\overset{\text{(A.1)}}{=} \psi(G^{-1}(\mu')) - \mathcal{T}_t(\mu', \mu). \end{aligned}$$

So, μ' is the unique maximizer in the variational formula of Step 1. In addition, according to Lemma 4.1, we have

$$p = \nabla\xi^*(\nabla\xi(p)) = \nabla\xi^*\left(\frac{q' - q}{t}\right) = \nabla\xi^*\left(\frac{G^{-1}(\mu') - q}{t}\right),$$

which characterizes p uniquely. □

Finally, we explain the difficulty that arises when one tries to generalize the arguments of [2] to $D > 1$. The argument relies on the fact that through the change of variable $q \mapsto \mathrm{Law}(q(U))$, the Parisi formula becomes a strictly concave optimization problem, and thus it admits only one maximizer. Let $\mathcal{P}^\uparrow(S^D_+)$ denote the image of $\mathcal{Q}$ by the map $p \mapsto \mathrm{Law}(q(U))$. When $D = 1$, $\mathcal{P}^\uparrow(S^D_+)$ is simply the set of probability measures on $\mathbb{R}_+$ which is a convex set (under the linear structure on the set of signed measures). When $D > 1$, $\mathcal{P}^\uparrow(S^D_+)$ is the set of probability measures with totally ordered support [20, Proposition 5.9] and it is not convex. For example when $D = 2$, letting

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

one can check that δ_A and δ_B belong to $\mathcal{P}^\uparrow(S^2_+)$ but not $(\delta_A + \delta_B)/2$. This prevents any easy generalization of the strict concavity argument from [2] to the case of vector spins (i.e. $D > 1$).

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